

# Edge Universality for Deformed Wigner Matrices

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We consider  $N \times N$  random matrices of the form  $H = W + V$  where  $W$  is a real symmetric Wigner matrix and  $V$  a random or deterministic, real, diagonal matrix whose entries are independent of  $W$ . We assume subexponential decay for the matrix entries of  $W$  and we choose  $V$  so that the eigenvalues of  $W$  and  $V$  are typically of the same order. For a large class of diagonal matrices  $V$  we show that the rescaled distribution of the extremal eigenvalues is given by the Tracy-Widom distribution  $F_1$  in the limit of large  $N$ . Our proofs also apply to the complex Hermitian setting, i.e., when  $W$  is a complex Hermitian Wigner matrix.

*AMS Subject Classification (2010):* 15B52, 60B20, 82B44

*Keywords:* Random matrix, Edge Universality

## 1. INTRODUCTION

It is widely believed that the behavior of the extremal eigenvalues of many random matrix ensembles is universal. This edge universality has been established for a large class of Wigner matrices: Let  $\mu_1$  denote the largest eigenvalue of a Wigner matrix of size  $N$ . The limiting distribution of  $\mu_1$  was identified for the Gaussian ensembles by Tracy and Widom [40, 41]. They proved that

$$\lim_{N \rightarrow \infty} \mathbb{P}(N^{2/3}(\mu_1 - 2) \leq s) = F_\beta(s), \quad (\beta \in \{1, 2\}), \quad (1.1)$$

$s \in \mathbb{R}$ , where the Tracy-Widom distribution functions  $F_\beta$  are described by Painlevé equations. The choice of  $\beta = 1, 2$  corresponds to the Gaussian Orthogonal/Unitary ensemble (GUE/GOE). The edge universality can also be extended to the  $k$  largest eigenvalues, where the joint distribution of the  $k$  largest eigenvalues can be written in terms of the Airy kernel, as first shown for the GUE/GOE in [21]. These results also hold for the  $k$  smallest eigenvalues.

Edge universality for Wigner matrices was first proved in [34] (see also [33]) for real symmetric and complex Hermitian ensembles with symmetric distributions. The symmetry assumption on the entries' distribution was partially removed in [29, 30]. Edge universality without any symmetry assumption was proved in [39] under the condition that the distribution of the matrix elements has subexponential decay and its first three moments match those of the Gaussian distribution, i.e., the third moment of the entries vanish. The vanishing third moment condition was removed in [19]. Recently, a necessary and sufficient condition on the entries' distribution for the edge universality of Wigner matrices was given in [27].

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\*Partially supported by National Research Foundation of Korea Grant 2011-0013474 and TJ Park Junior Faculty Fellowship.

†Supported by ERC Advanced Grant RANMAT, No. 338804.

In the present paper, we establish edge universality for deformed Wigner matrices: A deformed Wigner matrix,  $H$ , is an  $N \times N$  random matrix of the form

$$H = \lambda V + W, \quad (\lambda \in \mathbb{R}), \quad (1.2)$$

where  $V$  is a real, diagonal, random or deterministic matrix and  $W$  is a real symmetric or complex Hermitian Wigner matrix independent of  $V$ . The matrices are normalized so that the eigenvalues of  $V$  and  $W$  are order one. The “coupling” constant  $\lambda \in \mathbb{R}$  may depend on  $N$ , yet we will always assume that  $\lambda$  remains finite in the limit of large  $N$ . If the entries of  $V$  are random we may think of  $V$  as a “random potential”; if the entries of  $V$  are deterministic, matrices of the form (1.2) are sometimes referred to as “Wigner matrices with external source”. For  $W$  belonging to the GUE/GOE, the model (1.2) is often called the deformed GUE/GOE.

Assuming that the empirical eigenvalue distribution of  $V = \text{diag}(v_1, \dots, v_N)$ ,

$$\hat{\nu} := \frac{1}{N} \sum_{i=1}^N \delta_{v_i}, \quad (1.3)$$

converges weakly, respectively weakly in probability, to a non-random measure,  $\nu$ , it was shown in [28] that the empirical distribution of the eigenvalues of  $H$  converges weakly in probability to a deterministic measure which we refer to as the deformed semicircle law,  $\rho_{fc}$ . The deformed semicircle law  $\rho_{fc}$  depends on  $\nu$  and  $\lambda$ , and is thus in general distinct from Wigner’s semicircle law. For many choices of  $\nu$ , however, the deformed semicircle law  $\rho_{fc}$  has compact support and, similar to the standard semicircle law, exhibits a square-root type behavior at the endpoints of its support (see Lemma 4.3 for the precise statement). This suggests that the typical eigenvalue spacing at the spectral edge is of order  $N^{-2/3}$  as in the Wigner case and that the edge universality holds in the following sense. We assume that  $V$  is such that all eigenvalues of  $H$  stick to the support of the measure  $\rho_{fc}$ , i.e., that there are no “outliers” in the limit of large  $N$ . We further assume for simplicity that  $\rho_{fc}$  is supported on a single interval. Then the edge universality for deformed Wigner matrices states that there are  $\gamma_0 \equiv \gamma_0(N)$  and  $\hat{E}_+ \equiv \hat{E}_+(N)$ , such that the limiting distribution of the largest eigenvalue  $\mu_1$  of  $H$  satisfies

$$\lim_{N \rightarrow \infty} \mathbb{P}(\gamma_0 N^{2/3}(\mu_1 - \hat{E}_+) \leq s) = F_\beta(s), \quad (1.4)$$

where  $\gamma_0 > 0$  and  $\hat{E}_+ \in \mathbb{R}$  solely depend on  $\hat{\nu} \equiv \hat{\nu}(N)$  defined in (1.3) and the coupling constant  $\lambda$ . Further,  $\hat{E}_+$  converges in probability to the upper endpoint,  $E_+$ , of the deformed semicircle law  $\rho_{fc}$ . The scaling factor  $\gamma_0$  is order one and guarantees that the typical eigenvalue spacings at the edge of the rescaled matrix  $\gamma_0 H$  match those of the GOE/GUE up to negligible errors.

The deformed GUE for the special case when  $V$  has two eigenvalues  $\pm a$ , each with equal multiplicity, has been treated in a series of papers [7, 2, 8]. In this setting the local eigenvalue statistics at the edge can be obtained via the solution to a Riemann-Hilbert problem; see also [11] for the case when  $V$  has equispaced eigenvalues. For general  $V$ , the joint distribution of the eigenvalues of the deformed GUE can be expressed explicitly by the Brezin-Hikami/Johansson formula that admits an asymptotic analysis of the distribution of the extremal eigenvalues for various choices of  $V$  and ranges of  $\lambda$ ; see [22, 31, 9]. Once the edge universality has been established for the deformed GUE, it may be extended to complex Hermitian deformed Wigner matrices by appropriate modifications of the comparison methods introduced in [39] and in [19]. However, if the matrix  $W$  is real symmetric there is no explicit formula for the joint distribution of the eigenvalues available and the methods referred to above cannot be used to identify the Tracy-Widom distribution  $F_1$  in the real symmetric setting.

In the present paper, we establish the edge universality for real symmetric deformed Wigner matrices for a large class of  $V$  and wide ranges of  $\lambda$ ; see Theorem 2.8. In particular, we identify the Tracy-Widom distributions  $F_1$  as the limiting distributions of the extremal eigenvalues. Our proof also applies with minor modifications to the complex Hermitian setting, i.e., when  $W$  is a complex Hermitian matrix.

For the special case when the entries of  $V$  are independent and identically distributed (i.i.d.) random variables, for simplicity assumed to be bounded, we find that the limiting distribution of the largest rescaled eigenvalue of  $H$  is given by the convolution of the Tracy-Widom distribution and a centered Gaussian distribution with appropriately chosen variance depending on  $\lambda$ : The relative size of the Tracy-Widom part and the Gaussian part depends on the coupling constant  $\lambda$ ; the Gaussian part is negligible when  $\lambda \ll N^{-1/6}$ , whereas the Tracy-Widom component is dominated by the Gaussian if  $\lambda \gg N^{-1/6}$ . The transition from the Tracy-Widom to the Gaussian occurs at  $\lambda \sim N^{-1/6}$  as was pointed out first in [22] for the deformed GUE. Yet, the law of the eigenvalue spacing at the spectral edge is solely determined by the Tracy-Widom distribution for all finite  $\lambda$ . (See Theorem 2.11 for more detail.)

The main difficulty of the proof of our main results Theorem 2.8 and Theorem 2.11 lies in the proof of the Green function comparison theorem, Proposition 5.2. The Green function comparison method has shown to be very successful in proving the edge universality of Wigner matrices. However, the direct application of the conventional Green function

comparison uses Lindeberg's replacement strategy, which does not work for deformed Wigner matrices mainly due to the large diagonal elements. Simply put, as in the framework of the four-moment theorem in [38, 39], the usual method requires that the change of the averaged Green function from each replacement is  $o(N^{-2})$ , which is negligible since the number of such replacement is  $O(N^2)$ . On the other hand, for the deformed Wigner matrices with  $\lambda \sim 1$ , the replacement in the diagonal element causes an  $O(1)$  change in the averaged Green function, which is too large a change if the number of replacement steps is  $O(N)$ .

The main novelty of the present paper is a new approach to the Green function comparison theorem via Dyson Brownian motion (DBM). We estimate the change of the Green function along the flow of the DBM, which interpolates between the deformed Wigner matrix and the corresponding Gaussian ensemble. In other words, instead of converting the given random matrix entry by entry, we change all entries simultaneously, continuously. (See Section 6 for more detail.) The continuity of the DBM was used in [6] to compare the local eigenvalue statistics along the flow of the DBM for very short times. In our proof we follow the flow of the Green function over a time interval of order  $\log N$  during which it undergoes a change of order one. The continuous changes in the Green function can then be compensated by rescaling or "renormalizing" the matrix and the spectral parameter of the Green function. Such a proof of the Green function comparison requires, for  $\lambda \sim 1$ , some non-trivial estimates on functions of Green functions as is explained in Section 7. (See, e.g., (7.8) for such an estimate referred to as an "optical theorem" below.) For  $\lambda = 0$ , the presented method also yields, based on estimates in [19], a streamlined proof of the edge universality for Wigner matrices. (See Section 6.) For brevity we carry out the proof for real symmetric deformed Wigner matrices only, but the proof also applies with minor modifications to complex Hermitian deformed Wigner matrices.

Edge universality for deformed Wigner matrices may alternatively be studied via the local ergodicity of the DBM [17, 18]. This approach has been followed in [5] to prove the edge universality for generalized Wigner matrices. A basic ingredient of that proof is a global entropy estimate whose analogue version has been established for deformed Wigner matrices in Proposition 5.3 of [26] for some choices of  $V$ . Relying on this estimate, one can prove edge universality for deformed Wigner matrices following the lines of [5] (see Remark 2.9 in [26]). The advantage of the method presented in the present paper is that it is purely local: the only technical input is the local deformed semicircle law, i.e., estimates on the Green function on scale  $N^{-2/3}$ . (See Theorem 4.6 below.) Local laws for the deformed ensemble have been established in [24, 26]. However, in the proof presented in this paper these estimates are only needed at the edge of the spectrum and no further a priori control of the eigenvalues or Green function away from the edge is required. In particular, the method can also be used to study the extremal eigenvalues in a multi cut regime where the eigenvalues' limiting distribution is supported on several disjoint intervals. In such a setting the corresponding global entropy estimates in Proposition 5.3 of [26] were rather difficult to obtain. Another advantage of the method of the present paper is that it does not require that the eigenvalues evolve autonomously under DBM, i.e., that the stochastic differential equations for the eigenvalue and the eigenvectors decouple under DBM. The method can therefore also be applied to matrix ensembles for which the eigenvalues do not evolve autonomously under DBM.

This paper is organized as follows: In Section 2, we define the model precisely and introduce the main results of the paper. In Sections 3 and 4, we collect the tools and known results we need in the proof of the main results. In Section 5, we prove the main theorems using the Green function comparison theorem. In Sections 6-8, we explain the proof of the Green function comparison theorem. While the main ideas of the proof are rather nice and pleasant, the details of the proof of the Green function theorem include long explicit, but elementary, computations that can be found in the Appendices.

*Acknowledgements:* We thank Horng-Tzer Yau for numerous helpful discussions and remarks. We are also grateful to Paul Bourgade and László Erdős for helpful comments. We are grateful to Thomas Spencer for hospitality at the IAS where a major part of this research was conducted.

## 2. DEFINITION AND MAIN RESULT

### 2.1. Definition of the model.

**Definition 2.1.** Let  $W$  be an  $N \times N$  random matrix, whose entries  $(w_{ij})$  are independent, up to the symmetry constraint  $w_{ij} = w_{ji}$ , centered real random variables. We assume that the random variables  $(w_{ij})$  have variance  $1/N$  and have finite moments, uniformly in  $N$ ,  $i$  and  $j$ . More precisely, we assume that for each  $p \in \mathbb{N}$  there is a constant  $c_p$  such that

$$\mathbb{E}w_{ij} = 0, \quad \mathbb{E}w_{ij}^2 = \frac{1 + c_2\delta_{ij}}{N}, \quad \mathbb{E}|w_{ij}|^p \leq \frac{c_p}{N^{p/2}}, \quad (p \geq 3). \quad (2.1)$$

In case  $(w_{ij})$  are Gaussian random variables with  $c_2 = 1$ ,  $W$  belongs to the Gaussian orthogonal ensemble (GOE).

Let  $V = \text{diag}(v_i)$  be an  $N \times N$  diagonal, random or deterministic matrix, whose entries  $(v_i)$  are real-valued. We denote by  $\widehat{\nu}$  the empirical eigenvalue distribution of the diagonal matrix  $V = \text{diag}(v_i)$ ,

$$\widehat{\nu} := \frac{1}{N} \sum_{i=1}^N \delta_{v_i}. \quad (2.2)$$

**Assumption 2.2.** *There is a (non-random) compactly supported probability measure  $\nu$  and strictly positive constants  $\alpha_0$  and  $\beta_0$  such that the following holds. For any compact set  $\mathcal{D} \subset \mathbb{C}^+$  with  $\text{dist}(\mathcal{D}, \text{supp } \nu) > 0$ , there is  $C$  such that*

$$\mathbb{P} \left( \max_{z \in \mathcal{D}} \left| \int \frac{d\widehat{\nu}(v)}{v - z} - \int \frac{d\nu(v)}{v - z} \right| \leq CN^{-\alpha_0} \right) \geq 1 - N^{-\beta_0}, \quad (2.3)$$

for  $N$  sufficiently large.

Note that (2.3) implies that  $\widehat{\nu}$  converges to  $\nu$  in the weak sense as  $N \rightarrow \infty$ . Also note that the condition (2.3) holds for any  $0 < \alpha_0 < 1/2$  and any  $\beta_0 > 0$  if  $(v_i)$  are i.i.d. random variables.

We define the deformed Wigner matrix ensemble as follows:

**Definition 2.3.** *A deformed Wigner matrix of size  $N$  is an  $N \times N$  symmetric random matrix  $H$  that can be decomposed into*

$$H = (h_{ij}) := \lambda_0 V + W, \quad (2.4)$$

where  $W$  is a real symmetric Wigner matrix of size  $N$  and  $V = \text{diag}(v_i)$  is an  $N \times N$  real diagonal matrix. The entries of  $V$  can be random or deterministic. In case  $V$  is random, we assume that  $(v_i)$  are independent of  $(w_{ij})$ , yet  $(v_i)$  need not be independent among themselves. Finally,  $\lambda_0 \geq 0$  is a finite coupling constant.

Our second assumption on  $\widehat{\nu}$ ,  $\nu$  and  $\lambda_0$  guarantees that the limiting eigenvalue distribution of  $H$  is supported on a single interval and has a square root behavior at the two endpoints of the support. Sufficient conditions for this behavior have been presented in [32]. The assumption below also rules out the possibility that the matrix  $H$  has “outliers” in the limit of large  $N$ .

**Assumption 2.4.** *Let  $I_\nu$  be the smallest closed interval such that  $\text{supp } \nu \subseteq I_\nu$ . Then, there exists  $\varpi > 0$ , independent of  $N$ , such that*

$$\inf_{x \in I_\nu} \int \frac{d\nu(v)}{(v - x)^2} \geq (1 + \varpi) \lambda_0^2. \quad (2.5)$$

Moreover, let  $I_{\widehat{\nu}}$  be the smallest closed interval such that  $\text{supp } \widehat{\nu} \subseteq I_{\widehat{\nu}}$ . Then, we assume that there is a constant  $\beta_1 > 0$ , such that

$$\mathbb{P} \left( \inf_{x \in I_{\widehat{\nu}}} \int \frac{d\widehat{\nu}(v)}{(v - x)^2} \geq (1 + \varpi) \lambda_0^2 \right) \geq 1 - N^{-\beta_1}, \quad (2.6)$$

for  $N$  sufficiently large.

**Remark 2.5.** The left side of (2.5) may be infinite. In this case (2.5) should be understood in the sense that  $\lambda_0$  can be chosen as any finite positive number (independent of  $N$ ). Note that if (2.5) is satisfied for some  $\lambda \equiv \lambda_0$  and  $\nu$ , then it is also satisfied for all  $\lambda_0 \in [0, \lambda]$  for this  $\nu$ .

**Remark 2.6.** The coupling constant  $\lambda_0$  can be chosen to depend on  $N$ , as long as it stays bounded and converges sufficiently fast in the limit of large  $N$ . To simplify the exposition we only consider the case  $\lambda_0 = \sigma_0 N^{-\delta}$ , for some constants  $\delta \geq 0$  and  $\sigma_0 \geq 0$ , below.

We give two examples for which Assumption 2.4 is satisfied: We choose  $(v_i)$  to be i.i.d. random variables with law  $\nu$ .

- (1) Choosing  $\nu = \frac{1}{2}(\delta_{-1} + \delta_1)$ ,  $\lambda \geq 0$ , we have  $I_\nu = [-1, 1]$ . For  $\lambda < 1$ , one checks that there exist  $\varpi$  and  $\beta_1 > 0$  such that (2.5) and (2.6) are satisfied and that the deformed semicircle law is supported on a single interval with a square root type behavior at the edges. However, in case  $\lambda > 1$ , the deformed semicircle law is supported on two disjoint intervals. For more details see [7, 2, 8].
- (2) Let  $\nu$  to be a centered Jacobi measure of the form

$$\nu(v) = Z^{-1} (1 + v)^a (1 - v)^b d(v) \mathbb{1}_{[-1, 1]}(v), \quad (2.7)$$

where  $d \in C^1([-1, 1])$ , with  $d(v) > 0$ ,  $-1 < a, b < \infty$ , and  $Z$  a normalization constant. Then for  $a, b < 1$ , there is, for any  $\lambda \geq 0$ ,  $\varpi \equiv \varpi(\lambda) > 0$  and  $\beta_1 > 0$  such that (2.5) and (2.6) are satisfied. However, if  $a > 1$  or  $b > 1$  then (2.4) may not be satisfied for  $\lambda_0$  sufficiently large. In this setting the deformed semicircle law is still supported on a single interval, but the square root behavior at the edge may fail. We refer to [24, 25] for a detailed discussion.

**2.2. Deformed semicircle law.** The deformed semicircle law can be described in terms of the Stieltjes transform: For a (probability) measure  $\omega$  on the real line we define its Stieltjes transform,  $m_\omega$ , by

$$m_\omega(z) := \int \frac{d\omega(v)}{v - z}, \quad (z \in \mathbb{C}^+). \quad (2.8)$$

Note that  $m_\omega$  is an analytic function in the upper half plane and that  $\text{Im } m_\omega(z) \geq 0$ ,  $\text{Im } z > 0$ . Assuming that  $\omega$  is absolutely continuous with respect to Lebesgue measure, we can recover the density of  $\omega$  from  $m_\omega$  by the inversion formula

$$\omega(E) = \lim_{\eta \searrow 0} \frac{1}{\pi} \text{Im } m_\omega(E + i\eta), \quad (E \in \mathbb{R}). \quad (2.9)$$

We use the same symbols to denote measures and their densities.

Choosing  $\omega$  to be the standard semicircular law  $\rho_{sc}$ , the Stieltjes transform  $m_{sc} \equiv m_{\rho_{sc}}$  can be computed explicitly and one checks that  $m_{sc}$  satisfies the relation

$$m_{sc}(z) = \frac{-1}{m_{sc}(z) + z}, \quad \text{Im } m_{sc}(z) \geq 0, \quad (z \in \mathbb{C}^+). \quad (2.10)$$

The deformed semicircle law is conveniently defined through its Stieltjes transform. Let  $\nu$  be the limiting probability measure of Assumption 2.2. Then it is well-known [28] that the functional equation

$$m_{fc}(z) = \int \frac{d\nu(v)}{\lambda_0 v - z - m_{fc}(z)}, \quad \text{Im } m_{fc}(z) \geq 0, \quad (z \in \mathbb{C}^+), \quad (2.11)$$

has a unique solution that satisfies  $\limsup_{\eta \searrow 0} \text{Im } m(E + i\eta) < \infty$ , for all  $E \in \mathbb{R}$ . The deformed semicircle law, denoted by  $\rho_{fc}$ , is then defined through its density

$$\rho_{fc}(E) := \lim_{\eta \searrow 0} \frac{1}{\pi} \text{Im } m_{fc}(E + i\eta), \quad (E \in \mathbb{R}), \quad (2.12)$$

where  $m_{fc}$  is the solution to (2.11). The measure  $\rho_{fc}$  has been studied in detail in [4]. For example, it was shown there that the density  $\rho_{fc}$  is an analytic function inside the support of the measure. For our analysis the following result from [32, 24] is relevant.

**Lemma 2.7.** *Let  $\nu$  and  $\lambda_0$  satisfy (2.5) for some  $\varpi > 0$ . Then there are  $E_-, E_+ \in \mathbb{R}$ , such that  $\text{supp } \rho_{fc} = [E_-, E_+]$ . Moreover,  $\rho_{fc}$  has a strictly positive density on  $(E_-, E_+)$ .*

The measure  $\rho_{fc}$  is also called the additive free convolution of the semicircular law and, up to the rescaling by  $\lambda_0$ , the measure  $\nu$ . More generally, the additive free convolution of two (probability) measure  $\omega_1$  and  $\omega_2$ , usually denoted by  $\omega_1 \boxplus \omega_2$ , is defined as the distribution of the sum of two freely independent non-commutative random variables, having distributions  $\omega_1, \omega_2$  respectively; we refer to, e.g., [42, 1] for reviews. Similar to (2.11), the free convolution measure  $\omega_1 \boxplus \omega_2$  can be described in terms of a set of functional equations for the Stieltjes transforms; see [10, 3].

**2.3. Main result.** Let  $\mu_1^W$  be the largest eigenvalue of the Wigner matrix  $W$ . The edge universality for Wigner matrices asserts that

$$\lim_{N \rightarrow \infty} \mathbb{P} \left( N^{2/3} (\mu_1^W - 2) \leq s \right) = F_1(s), \quad (2.13)$$

where  $F_1$  is the Tracy-Widom distribution function for the Gaussian orthogonal ensembles. We remark that the Tracy-Widom distributions  $F_2$  and  $F_4$  arise as the limiting laws of the largest eigenvalues for the Gaussian unitary and Gaussian symplectic ensembles. Statement (2.13) holds true for the smallest eigenvalue  $\mu_N^W$  as well. We henceforth focus on the largest eigenvalues, the smallest eigenvalues can be dealt with in exactly the same way.

The edge universality for deformed real symmetric Wigner matrices, the main result of this paper, is as follows.

**Theorem 2.8.** *Let  $H = \lambda_0 V + W$  be a deformed Wigner matrix of the form (2.4), where  $W$  is a real symmetric Wigner matrix satisfying the assumptions in Definition 2.1,  $V$  is a real diagonal random or deterministic matrix satisfying Assumption 2.2 that is independent of  $W$ . Further, assume that  $V$  and  $\lambda_0 \geq 0$  satisfy Assumption 2.4. Let  $\mu_1$  be the largest eigenvalue of  $H$ .*

*Then, there exists  $\gamma_0 \equiv \gamma_0(N)$  and  $\hat{E}_+ \equiv \hat{E}_+(N)$  depending only on  $\lambda_0$  and  $\hat{\nu}$  such that the distribution of the rescaled largest eigenvalue converges to the Tracy-Widom distribution  $F_1$ , i.e.,*

$$\lim_{N \rightarrow \infty} \mathbb{P} \left( \gamma_0 N^{2/3} (\mu_1 - \hat{E}_+) \leq s \right) = F_1(s). \quad (2.14)$$

*Moreover,  $\hat{E}_+(N)$  converges in probability in the limit  $N \rightarrow \infty$  to  $E_+$ , the upper endpoint of the measure  $\rho_{fc}$ .*



**Remark 2.9.** A precise definition of  $\widehat{E}_+ \equiv \widehat{E}_+(N)$  is given in (4.10) and (4.16). As mentioned above  $\widehat{E}_+$  converges in probability to  $E_+$ , and we may replace  $\widehat{E}_+$  by  $E_+$  in (2.14) if the convergence is sufficiently fast. Note that the speed of convergence depends on the exponent  $\alpha_0$  in (2.3).

The normalization factor  $\gamma_0 \equiv \gamma_0(N)$  is given by

$$\gamma_0 = \left( - \int \frac{d\widehat{\nu}(v)}{(\lambda_0 v - \zeta)^3} \right)^{-1/3}, \quad (2.15)$$

where  $\zeta = \widehat{E}_+ + m_{fc}(\widehat{E}_+)$ ; see (4.7). For any  $\lambda_0$  such that Assumption 2.4 is satisfied, we have for some constant  $c$ , independent of  $N$ , that  $\zeta - \lambda_0 v > c > 0$ , for all  $v \in \text{supp } \widehat{\nu}$ . In particular,  $\gamma_0 = O(1)$ . It hence follows from Assumption 2.2 that we may replace  $\gamma_0$  by the  $N$ -independent quantity

$$\left( - \int \frac{d\nu(v)}{(\lambda_0 v - E_+ + m_{fc}(E_+))^3} \right)^{-1/3} \quad (2.16)$$

in (2.14).

**Remark 2.10.** Theorem 2.8 can be extended to correlation functions of extreme eigenvalues as follows: For any fixed  $k$ , the joint distribution function of the first  $k$  rescaled eigenvalues converges to that of the GOE, i.e., if we denote by  $\mu_1^{\text{GOE}} \geq \mu_2^{\text{GOE}} \geq \dots \geq \mu_N^{\text{GOE}}$  the eigenvalues of a GOE matrix, then

$$\begin{aligned} & \lim_{N \rightarrow \infty} \mathbb{P} \left( \gamma_0 N^{2/3} (\mu_1 - \widehat{E}_+) \leq s_1, \gamma_0 N^{2/3} (\mu_2 - \widehat{E}_+) \leq s_2, \dots, \gamma_0 N^{2/3} (\mu_k - \widehat{E}_+) \leq s_k \right) \\ &= \lim_{N \rightarrow \infty} \mathbb{P} \left( N^{2/3} (\mu_1^{\text{GOE}} - 2) \leq s_1, N^{2/3} (\mu_2^{\text{GOE}} - 2) \leq s_2, \dots, N^{2/3} (\mu_k^{\text{GOE}} - 2) \leq s_k \right). \end{aligned} \quad (2.17)$$

Our second result classifies the fluctuation of the largest eigenvalues of  $H = \lambda_0 V + W$  when the entries  $(v_i)$  of  $V$  are i.i.d. random variables that are independent of  $W$ .

**Theorem 2.11.** Let  $H = \lambda_0 V + W$  be a deformed Wigner matrix of the form (2.4), where  $W$  is a real symmetric Wigner matrix satisfying the assumptions in Definition 2.1 and  $V$  is a real diagonal random matrix independent of  $W$ . Assume that the entries of  $V$  are i.i.d. random variables with distribution  $\nu$ . Let  $m^{(k)}(\nu)$  denote the  $k$ -th central moment of  $\nu$ . (In particular,  $m^{(2)}(\nu)$  is the variance of  $\nu$ .) Let  $\mu_1$  be the largest eigenvalue of  $H$ .

Then, with  $E_+$  in Lemma 2.7, which depends only on  $\nu$  and  $\lambda_0$ , the following holds.

- i. Let  $\lambda_0 = \sigma_0 N^{-\delta}$ , for two constants  $\delta$  and  $\sigma_0$  satisfying  $1/6 < \delta$ ,  $\sigma_0 \geq 0$ . Then the distribution of the rescaled largest eigenvalue converges to the Tracy-Widom distribution, i.e.,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left( N^{2/3} (\mu_1 - E_+) \leq s \right) = F_1(s). \quad (2.18)$$

- ii. Let  $\lambda_0 = \sigma_0 N^{-1/6}$  for some constant  $\sigma_0 > 0$ . Then the distribution of the rescaled largest eigenvalue converges to the convolution of the Tracy-Widom distribution and the Gaussian distribution, i.e.,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left( N^{2/3} (\mu_1 - E_+) \leq s \right) = \mathbb{P}(X_F + X_\Phi \leq s), \quad (2.19)$$

where  $X_F$  and  $X_\Phi$  are independent random variables whose cumulative distribution functions are  $F_1$  and  $\Phi_{\sigma^2}$ , respectively. Here,  $\Phi_{\sigma^2}$  denotes the cumulative distribution function of a centered Gaussian distribution with variance  $\sigma^2 = \sigma_0^2 m^{(2)}(\nu)$ .

- iii. Let  $\lambda_0 = \sigma_0 N^{-\delta}$ , for two constants  $\delta$  and  $\sigma_0$  satisfying  $0 \leq \delta < 1/6$ ,  $\sigma_0 > 0$ . Then the distribution of the rescaled largest eigenvalue converges to the Gaussian distribution, i.e., there exist constants  $E_+, \sigma > 0$ , depending only on  $\lambda_0$  and  $\nu$ , such that

$$\lim_{N \rightarrow \infty} \mathbb{P} \left( N^{1/2} \lambda_0^{-1} (\mu_1 - E_+) \leq s \right) = \Phi_{\sigma^2}(s), \quad (2.20)$$

where the standard deviation  $\sigma$  is of order  $O(1)$ , satisfying

$$\sigma^2 = \lim_{N \rightarrow \infty} \lambda_0^{-2} (1 - m_{fc}(E_+))^2. \quad (2.21)$$

In particular, when  $\delta > 0$ , we have  $\sigma = m^{(2)}(\nu)$ .

Finally, let  $m_1(\nu)$  denote the mean of  $\nu$ . Then, the point  $E_+$  admits the asymptotic expansion

$$E_+ = 2 + \lambda_0 m_1(\nu) + \lambda_0^2 m^{(2)}(\nu) + \lambda_0^3 m^{(3)}(\nu) + \lambda_0^4 \left( m^{(4)}(\nu) - \frac{9(m^{(2)}(\nu))^2}{4} \right) + O(\lambda_0^5), \quad (2.22)$$

as  $\lambda_0 \rightarrow 0$ .

**Remark 2.12.** For the deformed GUE with deterministic  $V$ , Theorem 2.8 with deterministic potential has been obtained in [32] for rather general  $V$  and  $\lambda = O(1)$ . For the deformed GUE with random  $V$ , Theorem 2.8 has been established for some ranges of  $\lambda$  in [32]. The extension to all  $\lambda = O(1)$  was obtained in [9]. For random  $V$  with i.i.d. entries, statement *ii* of Theorem 2.11 at  $\lambda \sim N^{-1/6}$  has been established in [22] for the deformed GUE. For the deformed GOE with random  $V$ , partial results on the linear statistics of the eigenvalues at the edge have been obtained in [37] for  $\lambda = o(1)$ . Since very recently, there is a result [20] on bulk and edge universality for perturbations of Gaussian matrices under polynomials of matrices with an assumption on the asymptotic expansion of moments.

**Remark 2.13.** As remarked before, Assumption 2.4 insures that the deformed semicircle law  $\rho_{fc}$  has a square root decay at the edges. When Assumption 2.4 is not satisfied this may no longer be true and one expects a different edge behavior. Assuming that  $(v_i)$  are i.i.d. random variables with law given by a Jacobi measure as in (2.7) this has been studied in [25]. For example, when  $b > 1$  then there exists  $0 < \lambda_+ < \infty$  such that if  $\lambda_0 < \lambda_+$  then the Assumption 2.4 holds and the law of the rescaled largest eigenvalues converges to the Tracy-Widom distribution. When  $\lambda_0 > \lambda_+$ , the Assumption 2.4 is not satisfied, the deformed semicircle law  $\rho_{fc}$  does not have a square root behavior at the upper edge and the law of the rescaled largest eigenvalue converges to a Weibull distribution. Correspondingly, the eigenvectors associated to the largest eigenvalues are completely delocalized for  $\lambda_0 < \lambda_+$  (see [24]), while they are (partially) localized for  $\lambda_0 > \lambda_+$  (see [25]).

### 3. PRELIMINARIES

**3.1. Notations.** We introduce a notation for high-probability estimates which is suited for our purposes. A slightly different form was first used in [16].

**Definition 3.1.** *Let*

$$X = (X^{(N)}(u) : N \in \mathbb{N}, u \in U^{(N)}), \quad Y = (Y^{(N)}(u) : N \in \mathbb{N}, u \in U^{(N)}) \quad (3.1)$$

*be two families of nonnegative random variables where  $U^{(N)}$  is a possibly  $N$ -dependent parameter set. We say that  $Y$  stochastically dominates  $X$ , uniformly in  $U$ , if for all (small)  $\epsilon > 0$  and (large)  $D > 0$ ,*

$$\sup_{u \in U^{(N)}} \mathbb{P} \left[ X^{(N)}(u) > N^\epsilon Y^{(N)}(u) \right] \leq N^{-D}, \quad (3.2)$$

*for sufficiently large  $N \geq N_0(\epsilon, D)$ . If  $Y$  stochastically dominates  $X$ , uniformly in  $u$ , we write  $X \prec Y$ . If for some complex family  $X$  we have  $|X| \prec Y$  we also write  $X = \mathcal{O}(Y)$ . Further, if  $\mathbb{1}(\Xi)|X| \prec Y$  for some possibly  $N$ -dependent event  $\Xi^{(N)} \equiv \Xi$ , we also write  $X = \mathcal{O}_\Xi(Y)$ .*

For example, we have from (2.1) and Chebyshev's inequality that  $|h_{ij}| \prec \frac{1}{\sqrt{N}}$ . The relation  $\prec$  is a partial ordering: it is transitive and it satisfies the arithmetic rules of an order relation, e.g., if  $X_1 \prec Y_1$  and  $X_2 \prec Y_2$  then  $X_1 + X_2 \prec Y_1 + Y_2$  and  $X_1 X_2 \prec Y_1 Y_2$ .

We use the symbol  $O(\cdot)$  and  $o(\cdot)$  for the standard big-O and little-o notation. The notations  $O, o, \ll, \gg$ , refer to the limit  $N \rightarrow \infty$  unless stated otherwise. Here  $a \ll b$  means  $a = o(b)$ . We use  $c$  and  $C$  to denote positive constants that do not depend on  $N$ , usually with the convention  $c \leq C$ . Their value may change from line to line. We write  $a \sim b$ , if there is  $C \geq 1$  such that  $C^{-1}|b| \leq |a| \leq C|b|$ .

Finally, we use double brackets to denote index sets, i.e.,

$$\llbracket n_1, n_2 \rrbracket := [n_1, n_2] \cap \mathbb{Z},$$

for  $n_1, n_2 \in \mathbb{R}$ .

**3.2. Green function and minors.** Let  $A$  be an  $N \times N$  real symmetric matrix. The Green function or resolvent of  $A$  is defined as  $G_A(z) := (A - z)^{-1}$ ,  $z \in \mathbb{C}^+$ , and the averaged Green function of  $A$  is defined as  $m_A(z) := \frac{1}{N} \text{Tr } G_A(z)$ ,  $z \in \mathbb{C}^+$ . Bellow we often drop the subscript  $A$  and the argument  $z$  in  $G_A(z)$  and  $m_A(z)$ .

Let  $\mathbb{T} \subset \llbracket 1, N \rrbracket$ . Then we define  $A^{(\mathbb{T})}$  as the  $(N - |\mathbb{T}|) \times (N - |\mathbb{T}|)$  minor of  $A$  obtained by removing all columns and rows of  $A$  indexed by  $i \in \mathbb{T}$ . We do not change the names of the indices of  $A$  when defining  $A^{(\mathbb{T})}$ . More specifically, we define an operation  $\pi_i$ ,  $i \in \llbracket 1, N \rrbracket$ , on the probability space by

$$(\pi_i(A))_{kl} := \mathbb{1}(k \neq i) \mathbb{1}(l \neq i) h_{kl}. \quad (3.3)$$

Then, for  $\mathbb{T} \subset \llbracket 1, N \rrbracket$ , we set  $\pi_{\mathbb{T}} := \prod_{i \in \mathbb{T}} \pi_i$  and define

$$A^{(\mathbb{T})} := ((\pi_{\mathbb{T}}(A))_{ij})_{i,j \notin \mathbb{T}}. \quad (3.4)$$

The Green functions  $G^{(\mathbb{T})}$ , are defined in an obvious way using  $A^{(\mathbb{T})}$ . Moreover, we use the shorthand notations

$$\sum_i^{(\mathbb{T})} := \sum_{\substack{i=1 \\ i \notin \mathbb{T}}}^N, \quad \sum_{i \neq j}^{(\mathbb{T})} := \sum_{\substack{i=1, j=1 \\ i \neq j, i, j \notin \mathbb{T}}}^N, \quad (3.5)$$

abbreviate  $(i) = (\{i\})$ ,  $(\mathbb{T}i) = (\mathbb{T} \cup \{i\})$ . In Green function entries  $(G_{ij}^{(\mathbb{T})})$  we refer to  $\{i, j\}$  as *lower indices* and to  $\mathbb{T}$  as *upper indices*.

Finally, we set

$$m^{(\mathbb{T})} := \frac{1}{N} \sum_i^{(\mathbb{T})} G_{ii}^{(\mathbb{T})}. \quad (3.6)$$

Here, we use the normalization  $N^{-1}$ , instead  $(N - |\mathbb{T}|)^{-1}$ , since it is more convenient for our computations.

**3.3. Resolvent identities.** The next lemma collects the main identities between resolvent matrix elements of  $A$  and  $A^{(\mathbb{T})}$ .

**Lemma 3.2.** *Let  $A = A^*$  be an  $N \times N$  matrix. Consider the Green function  $G(z) := (A - z)^{-1}$ ,  $z \in \mathbb{C}^+$ . Then, for  $i, j, k, l \in \llbracket 1, N \rrbracket$ , the following identities hold:*

- Schur complement/Feshbach formula: For any  $i$ ,

$$G_{ii} = \frac{1}{a_{ii} - z - \sum_{k,l}^{(i)} a_{ik} G_{kl}^{(i)} a_{li}}. \quad (3.7)$$

- For  $i, j \neq k$ ,

$$G_{ij} = G_{ij}^{(k)} + \frac{G_{ik} G_{kj}}{G_{kk}}. \quad (3.8)$$

- For  $i \neq j$ ,

$$G_{ij} = -G_{ii} \sum_k^{(i)} a_{ik} G_{kj}^{(i)} = -G_{jj} \sum_k^{(j)} G_{ik}^{(j)} a_{kj}. \quad (3.9)$$

- For  $i \neq j$ ,

$$G_{ij} = -G_{ii} G_{jj}^{(i)} \left( a_{ij} - \sum_{k,l}^{(ij)} a_{ik} G_{kl}^{(ij)} a_{lj} \right). \quad (3.10)$$

For a proof we refer to, e.g., [13].

**3.4. Large deviation estimates.** Consider two families of random variables  $(X_i)$  and  $(Y_i)$ ,  $i \in \llbracket 1, N \rrbracket$ , satisfying

$$\mathbb{E} Z_i = 0, \quad \mathbb{E} |Z_i|^2 = 1, \quad \mathbb{E} |Z_i|^p \leq c_p, \quad (p \geq 3), \quad (3.11)$$

$Z_i = X_i, Y_i$ , for all  $p \in \mathbb{N}$  and some constants  $c_p$ , uniformly in  $i \in \llbracket 1, N \rrbracket$ . The following lemma, taken from [15], provides useful large deviation estimates.

**Lemma 3.3.** *Let  $(X_i)$  and  $(Y_i)$  be independent families of random variables and let  $(a_{ij})$  and  $(b_i)$ ,  $i, j \in \llbracket 1, N \rrbracket$ , be collections of complex numbers. Suppose that all entries  $(X_i)$  and  $(Y_i)$  are independent and satisfy (3.11). Then we have the bounds:*

$$\left| \sum_i b_i X_i \right| \prec \left( \sum_i |b_i|^2 \right)^{1/2}, \quad (3.12)$$

$$\left| \sum_i \sum_j a_{ij} X_i Y_j \right| \prec \left( \sum_{i,j} |a_{ij}|^2 \right)^{1/2}, \quad (3.13)$$

$$\left| \sum_{i \neq j} a_{ij} X_i X_j \right| \prec \left( \sum_{i \neq j} |a_{ij}|^2 \right)^{1/2}. \quad (3.14)$$



If the coefficients  $(a_{ij})$  and  $(b_i)$  depend on an additional parameter  $u$ , then all of these estimates are uniform in  $u$ , i.e. the threshold  $N_0 = N_0(\epsilon, D)$  in the definition of  $\prec$  depends only on the family  $c_p$  from (3.11); in particular,  $N_0$  does not depend on  $u$ .

#### 4. RENORMALIZATION OF THE MODEL

In this section, we rescale and “renormalize” the deformed Wigner matrix  $H = \lambda_0 V + W$  in order to setup later computations.

**4.1. Removal of the diagonal of  $W$  and fixing of  $V$ .** To simplify the notation in the upcoming sections, we replace the Wigner matrix  $W$  by

$$W - \text{diag}(w_{11}, \dots, w_{NN}), \quad (4.1)$$

i.e., we replace  $w_{ij}$  by  $w_{ij} - w_{ii}\delta_{ij}$ . With this modification we have  $H = (h_{ij})$ ,

$$h_{ii} = \lambda_0 v_i, \quad h_{ij} = w_{ij}, \quad (i \neq j). \quad (4.2)$$

By the next lemma, this replacement causes a negligible shift in the extremal eigenvalues of  $W$  or  $H = \lambda_0 V + W$  and we thus not explicitly display this modification in our notation.

**Lemma 4.1.** *Suppose that  $\lambda_0$ ,  $V$ , and  $W$  satisfy the assumptions in Theorem 2.8. Let  $H_1 = \lambda_0 V + W$  and  $H_2 = H_1 + C_2 \text{diag}(w_{11}, \dots, w_{NN})$  for some constant  $C_2$  independent of  $N$ . Further, let  $\mu_1(H_1)$  and  $\mu_1(H_2)$  be the largest eigenvalues of  $H_1$  and  $H_2$ , respectively. Then, there exists a constant  $\delta > 0$  such that, for any  $s \in \mathbb{R}$ , we have*

$$\begin{aligned} \mathbb{P}(N^{2/3}(\mu_1(H_1) - \widehat{E}_+) \leq s - N^{-\delta}) - N^{-\delta} &\leq \mathbb{P}(N^{2/3}(\mu_1(H_2) - \widehat{E}_+) \leq s) \\ &\leq \mathbb{P}(N^{2/3}(\mu_1(H_1) - \widehat{E}_+) \leq s + N^{-\delta}) + N^{-\delta}. \end{aligned} \quad (4.3)$$

The proof Lemma 4.1 follows almost verbatim from the proof of Lemma 3.5 in [27]. (See also Theorem 3.3 in [27]). We remark that the local deformed semicircle law for  $H_1$  and  $H_2$ , which is the key ingredient of the proof of Lemma 3.5 in [27], is given in Theorem 4.6 below.

To conveniently cope with the cases when  $(v_i)$  are random, respectively deterministic, we introduce an event  $\Xi$  on which the random variables  $(v_i)$  exhibit “typical” behavior. Recall that we denote by  $m_{\widehat{\nu}}$  and  $m_{\nu}$  the Stieltjes transforms of  $\widehat{\nu}$ , respectively  $\nu$ .

**Definition 4.2.** *Let  $\Xi \equiv \Xi(N)$  be an event such that the following holds on it:*

- (1) *There is a constant  $\alpha_0 > 0$  such that, for any compact set  $\mathcal{D} \subset \mathbb{C}^+$  with  $\text{dist}(\mathcal{D}, \text{supp } \nu) > 0$ , there is  $C$  such that*

$$|m_{\widehat{\nu}}(z) - m_{\nu}(z)| \leq CN^{-\alpha_0}, \quad (4.4)$$

*for  $N$  sufficiently large.*

- (2) *Recall the constant  $\varpi > 0$  and the intervals  $I_{\nu}$ ,  $I_{\widehat{\nu}}$  in Assumption 2.4. We have*

$$\inf_{x \in I_{\widehat{\nu}}} \int \frac{d\widehat{\nu}(v)}{(v-x)^2} \geq (1 + \varpi)\lambda_0^2, \quad \inf_{x \in I_{\nu}} \int \frac{d\nu}{(v-x)^2} \geq (1 + \varpi)\lambda_0^2, \quad (4.5)$$

*for  $N$  sufficiently large.*

In case  $(v_i)$  are deterministic,  $\Xi$  has full probability for  $N$  sufficiently large by Assumptions 2.2.

In the following we usually condition the random variables  $(v_i)$  on  $\Xi$ , i.e., we consider  $(v_i)$  as fixed and are such that (4.4) and (4.5) hold.

**4.2. Rescaling of  $H$ .** Let  $\lambda \in [0, \lambda_0]$ . In order to compare the local edge statistics of  $H = \lambda V + W$  with the local GOE edge statistics, it is natural to rescale  $H$  in such a way that the typical size of the eigenvalue spacing at the upper edge of the rescaled matrix match those of  $W$ . Put slightly differently, we can find  $\gamma \in \mathbb{R}$ , depending on  $\lambda_0$ , such that the eigenvalue gaps of  $\widetilde{H} := \gamma H$  typically agree for large  $N$  with the gaps predicted by the Tracy-Widom distribution.

The scaling factor  $\gamma$  may be constructed as follows. For any  $\lambda \in [0, \lambda_0]$ , let  $\zeta \equiv \zeta(\lambda)$  be the largest solution to

$$\int \frac{d\widehat{\nu}(v)}{(\lambda v - \zeta)^2} = 1. \quad (4.6)$$

We note that, for  $\widehat{\nu}$  satisfying (4.5), such  $\zeta$  exists for all  $\lambda \leq \lambda_0$ . Note that  $\zeta$  depends on  $\lambda$  and the measure  $\widehat{\nu}$ . We then define the scaling factor  $\gamma$  by

$$\gamma \equiv \gamma(\lambda) := \left( - \int \frac{d\widehat{\nu}(v)}{(\lambda v - \zeta)^3} \right)^{-1/3}, \quad \gamma_0 := \gamma(\lambda_0). \quad (4.7)$$

It follows from (4.6) and Hölder's inequality that  $0 < \gamma \leq 1$ . Note that for  $\lambda = 0$ , we have  $\zeta = 1$  and  $\gamma = 1$ .

We now set

$$\tilde{H} := \gamma H = \gamma(\lambda V + W), \quad (4.8)$$

and define the Green function, respectively averaged Green function, of  $\tilde{H}$  by

$$G_{\tilde{H}}(z) := \frac{1}{\gamma(\lambda V + W) - z}, \quad m_{\tilde{H}}(z) := \frac{1}{N} \text{Tr } G_{\tilde{H}}(z), \quad (z \in \mathbb{C}^+).$$

Next, we define  $\hat{m}_{fc}$  as the solution to the equation

$$\hat{m}_{fc}(z) = \int \frac{d\hat{\nu}(v)}{\lambda\gamma v - z - \gamma^2 \hat{m}_{fc}(z)}, \quad \text{Im } \hat{m}_{fc}(z) \geq 0, \quad (z \in \mathbb{C}^+). \quad (4.9)$$

Following the arguments in Subsection 2.2,  $\hat{m}_{fc}(z)$  defines a probability measure  $\hat{\rho}_{fc}$  whose density is given by  $\hat{\rho}_{fc}(E) = \frac{1}{\pi} \lim_{\eta \searrow 0} \text{Im } \hat{\rho}_{fc}(E + i\eta)$ ,  $E \in \mathbb{R}$ . For simplicity, we omit the  $\lambda$ - and  $\gamma$ -dependences of  $\hat{m}_{fc}$  and  $\hat{\rho}_{fc}$  from the notation. Note that the measure  $\hat{\rho}_{fc}$  depends on  $N$  through  $\hat{\nu}$ . However, this being the main point here,  $\hat{\rho}_{fc}$  does not depend on the matrix  $W$  in any way. On the event  $\Xi$  of typical realizations of  $(v_i)$ ,  $\hat{\rho}_{fc}$  enjoys the following properties:

**Lemma 4.3.** *On  $\Xi$  the following holds. There exist  $\hat{L}_-, \hat{L}_+ \in \mathbb{R}$ , with  $\hat{L}_- < \hat{L}_+$ , such that*

$$\text{supp } \hat{\rho}_{fc} = [\hat{L}_-, \hat{L}_+]. \quad (4.10)$$

Denoting by  $\kappa_E$  the distance to the endpoints of the support of  $\rho_{fc}$ , i.e.,

$$\kappa_E := \min\{|E - \hat{L}_-|, |E - \hat{L}_+|\}, \quad (4.11)$$

we have

$$C^{-1} \sqrt{\kappa_E} \leq \hat{\rho}_{fc}(E) \leq C \sqrt{\kappa_E}, \quad (E \in [\hat{L}_-, \hat{L}_+]), \quad (4.12)$$

for some constant  $C \geq 1$ , which can be chosen uniformly in  $\lambda \in [0, \lambda_0]$ . In particular, we have

$$\hat{\rho}_{fc}(E) = \frac{1}{\pi} \sqrt{\kappa_E} (1 + O(\kappa_E)), \quad (4.13)$$

as  $E \rightarrow \hat{L}_+$ ,  $E \leq \hat{L}_+$ .

Further, we have the following estimates for the imaginary part of  $\hat{m}_{fc}$ :

(1) For  $z = \hat{L}_+ - \kappa + i\eta$ , with  $0 \leq \kappa \leq \hat{L}_+$  and  $0 < \eta \leq 2$ , there exists a constant  $C \geq 1$  such that

$$C^{-1} \sqrt{\kappa_E + \eta} \leq \text{Im } \hat{m}_{fc}(z) \leq C \sqrt{\kappa_E + \eta}. \quad (4.14)$$

(2) For  $z = \hat{L}_+ + \kappa + i\eta$ , with  $0 \leq \kappa \leq 1$  and  $0 < \eta \leq 2$ , there exists a constant  $C \geq 1$  such that

$$C^{-1} \frac{\eta}{\sqrt{\kappa_E + \eta}} \leq \text{Im } \hat{m}_{fc}(z) \leq C \frac{\eta}{\sqrt{\kappa_E + \eta}}. \quad (4.15)$$

Moreover, all constants can be chosen uniformly in  $\lambda \in [0, \lambda_0]$ .

The proof of Lemma 4.3 can be found in [26]. Returning to the introductory remarks of this subsection, we emphasize (4.13): The scaling factor  $\gamma$  has been chosen such that (4.13) holds for all  $\lambda \in [0, \lambda_0]$ , i.e., up to a global shift, the measure  $\hat{\rho}_{fc}^0$  exhibits a square root decay at the upper edge with the same rate as the standard semicircle law  $\rho_{sc}$ .

**Remark 4.4.** Let  $\hat{m}_{fc}^0$  be the solution to the equation

$$\hat{m}_{fc}^0(z) = \int \frac{d\hat{\nu}(v)}{\lambda v - z - \hat{m}_{fc}^0(z)}, \quad \text{Im } \hat{m}_{fc}^0(z) \geq 0, \quad (z \in \mathbb{C}^+),$$

and  $\hat{\rho}_{fc}^0$  the probability measure whose density is given by  $\hat{\rho}_{fc}^0(E) := \frac{1}{\pi} \lim_{\eta \searrow 0} \text{Im } \hat{\rho}_{fc}^0(E + i\eta)$ ,  $E \in \mathbb{R}$ . As in Lemma 4.3 we find that there are  $\hat{E}_-, \hat{E}_+ \in \mathbb{R}$  such that  $\text{supp } \hat{\rho}_{fc}^0 = [\hat{E}_-, \hat{E}_+]$  and that  $\hat{\rho}_{fc}^0$  has a strictly positive density in  $(\hat{E}_-, \hat{E}_+)$ . By definition, it is obvious that

$$\hat{E}_+ = \gamma^{-1} \hat{L}_+. \quad (4.16)$$

**Remark 4.5.** The scaling factor  $\gamma$  defined in (4.7) satisfies the relation

$$\gamma = \left( - \int \frac{d\widehat{\nu}(v)}{(\lambda\gamma v - \tau)^3} \right)^{-1/6}, \quad (4.17)$$

where  $\tau = \widehat{L}_+ + \gamma^2 m_{fc}(\widehat{L}_+)$ .

To see that (4.7) implies (4.17), we note that  $\widehat{m}_{fc}^0(\widehat{E}_+) = \gamma \widehat{m}_{fc}(\widehat{L}_+)$ , hence  $\zeta$  in (4.6) satisfies  $\zeta = \widehat{E}_+ + \widehat{m}_{fc}^0(\widehat{E}_+)$ . This also shows that

$$\int \frac{d\widehat{\nu}(v)}{(\lambda\gamma v - \widehat{L}_+ - \gamma^2 \widehat{m}_{fc}(\widehat{L}_+))^2} = \frac{1}{\gamma^2} \int \frac{d\widehat{\nu}(v)}{(\lambda v - \widehat{E}_+ - \widehat{m}_{fc}^0(\widehat{E}_+))^2} = \frac{1}{\gamma^2} \int \frac{d\widehat{\nu}(v)}{(\lambda v - \zeta)^2} = \frac{1}{\gamma^2}. \quad (4.18)$$

We define  $\tau$  by

$$\tau = \widehat{L}_+ + \gamma^2 \widehat{m}_{fc}(\widehat{L}_+) = \gamma \zeta, \quad (4.19)$$

and find that

$$\int \frac{d\widehat{\nu}(v)}{(\lambda\gamma v - \tau)^3} = \frac{1}{\gamma^3} \int \frac{d\widehat{\nu}(v)}{(\lambda v - \zeta)^3} = -\frac{1}{\gamma^6}.$$

This proves (4.17).

Next, we collect estimates on the Green function of  $\widetilde{H}$ . Fix a small  $\xi > 0$  and define the domain

$$\mathcal{D}_\xi := \{z = E + i\eta \in \mathbb{C}^+ : 0 \leq |E| \leq \lambda_0 + 1, N^{-1+\xi} \leq \eta \leq 3\}. \quad (4.20)$$

We also introduce the control parameter

$$\Pi(z) := \sqrt{\frac{\operatorname{Im} \widehat{m}_{fc}(z)}{N\eta}} + \frac{1}{N\eta}. \quad (4.21)$$

The next theorem is the local deformed semicircle law for  $\widetilde{H}$ , which was established in Theorem 3.3 of [26].

**Theorem 4.6** (Local deformed semicircle law). *On  $\Xi$ , the following holds true. For any small fixed  $\xi > 0$ , we have*

$$|m_{\widetilde{H}}(z) - \widehat{m}_{fc}(z)| \prec \frac{1}{N\eta}, \quad \max_{i \neq j} |(G_{\widetilde{H}})_{ij}(z)| \prec \Pi(z), \quad (4.22)$$

uniformly in  $z \in \mathcal{D}_\xi$  and  $\lambda \in [0, \lambda_0]$ . Further, setting  $g_i(z) := (\lambda\gamma v_i - z - \gamma^2 \widehat{m}_{fc}(z))^{-1}$ , we also have

$$\max_i |(G_{\widetilde{H}})_{ii}(z) - g_i(z)| \prec \Pi(z), \quad (4.23)$$

uniformly in  $z \in \mathcal{D}_\xi$  and  $\lambda \in [0, \lambda_0]$ . In particular, we have  $|(G_{\widetilde{H}})_{ii}(z)| \prec 1$ .

The following lemma gives a rigidity estimate on the eigenvalue location of  $\widetilde{H}$ . We denote by  $\widetilde{\mu}_1 \geq \widetilde{\mu}_2 \geq \dots \geq \widetilde{\mu}_N$  the eigenvalues of  $\widetilde{H}$  in descending order. Define the “classical” location,  $\gamma_k$ , of the  $k$ -th eigenvalue of  $\widetilde{H}$  by

$$\int_{-\infty}^{\gamma} \widehat{\rho}_{fc}(x) dx = \frac{k}{N}, \quad (k \in \llbracket 1, N \rrbracket). \quad (4.24)$$

**Lemma 4.7** (Rigidity of eigenvalues). *On  $\Xi$ , we have*

$$|\widetilde{\mu}_k - \gamma_k| \prec N^{-2/3} \left( \frac{1}{k} \right)^{1/3}, \quad (4.25)$$

uniformly in  $\lambda \in [0, \lambda_0]$ , where we have set  $\widehat{k} := \min\{k, N - k\}$ .

Lemma 4.7 follows from Theorem 4.6 by an application of the Helffer-Sjöstrand calculus. The proof of Lemma 5.1 in [19] for (generalized) Wigner matrices applies ad verbum to deformed Wigner matrices.

Alluding once more to the introductory remarks of the present subsection, we remark that the classical locations  $(\gamma_\alpha)$  depend on  $\lambda$ . Yet, close to the upper edge, i.e.,  $\alpha \ll N^{1/3}$ , the gaps  $\gamma_{\alpha+1} - \gamma_\alpha$  are essentially independent of  $\lambda$  as follow from (4.13). From Lemma 4.7, we can extend this conclusion to the eigenvalue gaps  $\widetilde{\mu}_{\alpha+1} - \widetilde{\mu}_\alpha$  at the upper edge on  $\Xi$  for  $N$  sufficiently large.

**Remark 4.8.** The local law in Theorem 4.6 and the rigidity result in Lemma 4.7 are stronger than the corresponding results in [24]. The improvement is based on fixing the diagonal element  $(v_i)$ . See Theorem 2.12, Remark 2.12, and Remark 2.14 in [24] for more discussion. In fact, the estimates in Theorem 4.6 and Lemma 4.7 are essentially optimal up to corrections  $N^\epsilon$ .

## 5. PROOF OF MAIN RESULTS

**5.1. Density of states and the averaged Green function.** We follow the proof of the edge universality in [19, 14]. Recall that  $\tilde{\mu}_1 \geq \tilde{\mu}_2 \geq \dots \geq \tilde{\mu}_N$  denote the eigenvalues of  $\tilde{H}$  and  $\hat{L}_+$  is the upper edge of  $\text{supp } \hat{\rho}_{fc}$ . Recall the event  $\Xi$  in Definition 4.2. From Lemma 4.7, we find that

$$|\tilde{\mu}_1 - \hat{L}_+| \prec N^{-2/3},$$

on  $\Xi$ . Thus, we may assume in (2.14) that  $s \prec 1$ .

Fix  $E_*$  such that

$$E_* - \hat{L}_+ \prec N^{-2/3}, \quad \mathbb{1}(\mu_1 - E_* > 0) \prec 0.$$

We note that the choice of  $E_*$  guarantees that the event  $\mu_1 > E_*$  is negligible. For  $E$  satisfying

$$|E - \hat{L}_+| \prec N^{-2/3}, \quad (5.1)$$

we let

$$\chi_E := \mathbb{1}_{[E, E_*]}.$$

We also define the Poisson kernel,  $\theta_\eta$ , for  $\eta > 0$ , by

$$\theta_\eta(x) := \frac{\eta}{\pi(x^2 + \eta^2)} = \frac{1}{\pi} \text{Im} \frac{1}{x - i\eta}.$$

Introduce a smooth cutoff function  $K : \mathbb{R} \rightarrow \mathbb{R}$  satisfying

$$K(x) = \begin{cases} 1 & \text{if } x \leq 1/9, \\ 0 & \text{if } x \geq 2/9. \end{cases} \quad (5.2)$$

Let  $\mathcal{N}(E_1, E_2)$  be the number of the eigenvalues in  $(E_1, E_2]$ , i.e.,

$$\mathcal{N}(E_1, E_2) := |\{\alpha : E_1 < \tilde{\mu}_\alpha \leq E_2\}|,$$

and define the density of states in the interval  $[E_1, E_2]$  by

$$\mathbf{n}(E_1, E_2) := \frac{1}{N} \mathcal{N}(E_1, E_2).$$

In order to estimate  $\mathbb{P}(\tilde{\mu}_1 \leq E)$ , we consider the following approximation:

$$\mathbb{P}(\tilde{\mu}_1 \leq E) = \mathbb{E}K(\mathcal{N}(E, \infty)) \simeq \mathbb{E}K(\mathcal{N}(E, E_*)) \simeq \mathbb{E}K\left(N \int_E^{E_*} \text{Im } m(y + i\eta) dy\right), \quad (5.3)$$

with  $\eta \sim N^{-2/3-\epsilon'}$ , for some small  $\epsilon' > 0$ . The first approximation in (5.3) follows from Lemma 4.7, the rigidity of the eigenvalues, and the second from

$$\mathcal{N}(E, E_*) = \text{Tr } \chi_E(H) \simeq \text{Tr } \chi_E * \theta_\eta(H) = \frac{1}{\pi} N \int_E^{E_*} \text{Im } m(y + i\eta) dy.$$

The following lemma shows that the approximations in (5.3) indeed hold.

**Lemma 5.1.** *Suppose that  $E$  satisfies (5.1). Let  $K$  be a smooth function satisfying (5.2). For  $\epsilon > 0$ , let  $\ell := \frac{1}{2}N^{-2/3-\epsilon}$  and  $\eta := N^{-2/3-9\epsilon}$ . Then, for any sufficiently small  $\epsilon > 0$  and any (large)  $D > 0$ , we have*

$$\text{Tr}(\chi_{E+\ell} * \theta_\eta(H)) - N^{-\epsilon} \leq \mathbf{n}(E, \infty) \leq \text{Tr}(\chi_{E-\ell} * \theta_\eta(H)) + N^{-\epsilon} \quad (5.4)$$

and

$$\mathbb{E}K(\text{Tr}(\chi_{E-\ell} * \theta_\eta(H))) \leq \mathbb{P}(\tilde{\mu}_1 \leq E) \leq \mathbb{E}K(\text{Tr}(\chi_{E+\ell} * \theta_\eta(H))) + N^{-D}, \quad (5.5)$$

for any sufficiently large  $N \geq N_0(\epsilon, D)$ .

*Proof.* We may follow the proof of Corollary 6.2 of [19]. (See also Lemma 6.5 of [14].) Note that the estimates on  $|m(E + i\ell) - m_{fc}(E + i\ell)|$  and  $\text{Im } m_{fc}(E - \kappa + i\ell)$ , which replace similar estimates with respect to  $m_{sc}$  in the proof of Corollary 6.2 in [19], are already proved in Lemma 4.3 and Theorem 4.6.  $\square$

**5.2. Green function comparison and proof of Theorem 2.8.** We now prove the main result of the paper using the following proposition, which compares the right side of (5.3) and the corresponding expectation with respect to the Wigner matrix  $W$ . Recall that the averaged Green function of  $\tilde{H}$  is defined by

$$m_{\tilde{H}}(z) := \frac{1}{N} \operatorname{Tr}(\tilde{H} - z)^{-1}, \quad (z \in \mathbb{C}^+).$$

Let  $W^{\text{GOE}}$  be a standard GOE matrix which is independent of  $V$  and  $W$ . We define the averaged Green function of  $W^{\text{GOE}}$  by

$$m^{\text{GOE}}(z) := \frac{1}{N} \operatorname{Tr}(W^{\text{GOE}} - z)^{-1}, \quad (z \in \mathbb{C}^+).$$

**Proposition 5.2** (Green function comparison). *Let  $\epsilon > 0$  and set  $\eta = N^{-2/3-\epsilon}$ . Let  $E_1, E_2 \in \mathbb{R}$  satisfy*

$$|E_1 - 2| \leq N^{-2/3+\epsilon}, \quad |E_2 - 2| \leq N^{-2/3+\epsilon}. \quad (5.6)$$

*Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function satisfying*

$$\max_x |F^{(\ell)}(x)|(|x| + 1)^{-C} \leq C, \quad \ell = 1, 2, 3, 4. \quad (5.7)$$

*Then, there exists a constant  $C' > 0$  such that, for any sufficiently large  $N$  and for any sufficiently small  $\epsilon > 0$ , we have that, on  $\Xi$ ,*

$$\left| \mathbb{E} F \left( N \int_{E_1}^{E_2} \operatorname{Im} m_{\tilde{H}}(x + \hat{L}_+ - 2 + i\eta) dx \right) - \mathbb{E} F \left( N \int_{E_1}^{E_2} \operatorname{Im} m^{\text{GOE}}(x + i\eta) dx \right) \right| \leq N^{-1/6+C'\epsilon}, \quad (5.8)$$

*where the expectation  $\mathbb{E}$  is with respect to  $W$ .*

We prove Proposition 5.2 in the Section 6.

**Remark 5.3.** Proposition 5.2 can be extended as follows: Let  $\epsilon > 0$  and set  $\eta = N^{-2/3-\epsilon}$ . Let  $E_0, E_1, \dots, E_k \in \mathbb{R}$  satisfy

$$|E_0 - 2| \leq N^{-2/3+\epsilon}, \quad |E_1 - 2| \leq N^{-2/3+\epsilon}, \quad \dots, \quad |E_k - 2| \leq N^{-2/3+\epsilon}.$$

Let  $F : \mathbb{R}^k \rightarrow \mathbb{R}$  be a smooth function satisfying

$$\max_x |F^{(\ell)}(x)|(|x| + 1)^{-C} \leq C, \quad \ell = 1, 2, 3, 4.$$

Then, there exists a constant  $C' > 0$  such that, for any sufficiently large  $N$  and for any sufficiently small  $\epsilon > 0$ , we have on  $\Xi$  that

$$\left| \mathbb{E} F \left( N \int_{E_1}^{E_0} \operatorname{Im} m_{\tilde{H}}(x + \hat{L}_+ - 2 + i\eta) dx, \dots, N \int_{E_k}^{E_0} \operatorname{Im} m_{\tilde{H}}(x + \hat{L}_+ - 2 + i\eta) dx \right) - \mathbb{E} F \left( N \int_{E_1}^{E_0} \operatorname{Im} m^{\text{GOE}}(x + i\eta) dx, \dots, N \int_{E_k}^{E_0} \operatorname{Im} m^{\text{GOE}}(x + i\eta) dx \right) \right| \leq N^{-1/6+C'\epsilon}. \quad (5.9)$$

The proof of (5.9) is similar to that of Proposition 5.2 and will be omitted. Assuming the validity of the proposition, we now prove the main result.

*Proof of Theorem 2.8.* Recall that we denote by  $\mu_1^{\text{GOE}} \geq \mu_2^{\text{GOE}} \geq \dots \geq \mu_N^{\text{GOE}}$  the eigenvalues of  $W^{\text{GOE}}$ . Since  $\mathbb{P}(\Xi) \rightarrow 1$  as  $N \rightarrow \infty$  by assumption, we may assume that  $V$  is fixed and condition on  $\Xi$ . Thus, to prove (2.14), it suffices to establish

$$\mathbb{P}[N^{2/3}(\mu_1^{\text{GOE}} - 2) \leq s] - N^{-\phi} < \mathbb{P}[N^{2/3}(\tilde{\mu}_1 - \hat{L}_+) \leq s] < \mathbb{P}[N^{2/3}(\mu_1^{\text{GOE}} - 2) \leq s] + N^{-\phi}, \quad (5.10)$$

for some  $\phi > 0$ .

Fix  $s \prec 1$  and let  $E := \hat{L}_+ + sN^{-2/3}$ . Let  $\ell := \frac{1}{2}N^{-2/3-\epsilon}$  and  $\eta := N^{-2/3-9\epsilon}$ . For any sufficiently small  $\epsilon > 0$ , we have from Lemma 5.1 that

$$\mathbb{P}(\tilde{\mu}_1 \leq E) \geq \mathbb{E}[K(\operatorname{Tr}(\chi_{E-\ell} * \theta_\eta(H)))] .$$

From Proposition 5.2, we find that

$$\mathbb{E}[K(\operatorname{Tr}(\chi_{E-\ell} * \theta_\eta(H)))] \geq \mathbb{E}\left[K\left(\operatorname{Tr}\left(\chi_{E-(\hat{L}_+-2)-\ell} * \theta_\eta(W^{\text{GOE}})\right)\right)\right] - N^{-\phi},$$

for some  $\phi > 0$ . Finally, we have from Corollary 6.2 of [19] that

$$\mathbb{E}\left[K\left(\operatorname{Tr}\left(\chi_{E-(\hat{L}_+-2)-\ell} * \theta_\eta(W^{\text{GOE}})\right)\right)\right] \geq \mathbb{P}\left(\mu_1^{\text{GOE}} \leq E - (\hat{L}_+ - 2)\right) - N^{-\phi}.$$

Altogether, we have shown that

$$\mathbb{P}(\tilde{\mu}_1 \leq E) \geq \mathbb{P}(\mu_1^{\text{GOE}} \leq E - (\hat{L}_+ - 2)) - 2N^{-\phi},$$

which proves the first inequality of (5.10). The second inequality can be proved similarly.

To complete the proof of the desired theorem, we notice that it was proved in Lemma C.1 of [25] that there exists a random variable  $X \equiv X(N)$ , which converges to the Gaussian random variable with mean 0 and variance  $N^{-1}(1 - (m_{fc}(E_+))^2)$ , satisfying

$$\hat{E}_+ - E_+ = X + \mathcal{O}(N^{-1}),$$

on  $\Xi$ . In the proof of Theorem 2.11, we will show that  $\text{Var}(X) \sim N^{-1}\lambda_0^2$ , which implies that  $\hat{E}_+$  converges in probability to  $E_+$ .  $\square$

Using the general form of the Green function comparison as in (5.9), we can prove (2.10) in a similar manner.

**5.3. Proof of Theorem 2.11.** We next prove Theorem 2.11. Recall that  $\rho_{fc}$  denotes the deformed semicircle measure, whose Stieltjes transform is denoted by  $m_{fc}$ .

*Proof of Theorem 2.11.* For simplicity, assume that  $\nu$  is centered; the proof is essentially the same even if  $\nu$  is not centered. Recall that we denote by  $m^{(n)}(\nu)$  the  $n$ -th central moment of  $\nu$ . We notice that  $\text{supp } \rho_{fc} = [E_-, E_+]$ , for some  $E_- < 0 < E_+$ ; see Lemma 2.7. As pointed out in the proof of Theorem 2.8, there exists a random variable  $X$ , which converges to the Gaussian random variable with mean 0 and variance  $N^{-1}(1 - (m_{fc}(E_+))^2)$ , satisfying

$$\hat{E}_+ - E_+ = X + \mathcal{O}(N^{-1}),$$

on  $\Xi$ . Let

$$\vartheta := E_+ + m_{fc}(E_+) = E_+ + \int \frac{d\nu(v)}{\lambda_0 v - \vartheta}.$$

It was shown in [24] that  $\vartheta$  is the solution to the equation

$$\int \frac{d\nu(v)}{(\lambda_0 v - \vartheta)^2} = 1, \quad \vartheta \geq 0, \quad (5.11)$$

and that there exists a constant  $c > 0$ , independent of  $N$ , such that  $\vartheta - \lambda_0 v > c$  for any  $v \in \text{supp } \nu$ .

We first consider the case  $\lambda_0 \ll 1$ . Expanding (5.11) in terms of  $\lambda_0$ , we obtain

$$1 = \int \frac{d\nu(v)}{(\lambda_0 v - \vartheta)^2} = \int d\nu(v) \left( \frac{1}{\vartheta^2} + \frac{2\lambda_0 v}{\vartheta^3} + \frac{3\lambda_0^2 v^2}{\vartheta^4} + \mathcal{O}(\lambda_0^3) \right) = \frac{1}{\vartheta^2} + \frac{3\lambda_0^2 m^{(2)}(\nu)}{\vartheta^4} + \mathcal{O}(\lambda_0^3), \quad (5.12)$$

where we used that  $\nu$  is centered and has variance  $m^{(2)}(\nu)$ . We thus get  $\vartheta^2 = 1 + \mathcal{O}(\lambda_0^2)$  and, by putting it back into (5.12),

$$\vartheta = 1 + \frac{3\lambda_0^2 m^{(2)}(\nu)}{2} + \mathcal{O}(\lambda_0^3).$$

We now have

$$m_{fc}(E_+) = \int \frac{d\nu(v)}{\lambda_0 v - \vartheta} = -\frac{1}{\vartheta} - \frac{\lambda_0^2 m^{(2)}(\nu)}{\vartheta^3} + \mathcal{O}(\lambda_0^3) = -1 + \frac{\lambda_0^2 m^{(2)}(\nu)}{2} + \mathcal{O}(\lambda_0^3),$$

and thus

$$N\text{Var}(X) = 1 - (m_{fc}(E_+))^2 = \lambda_0^2 m^{(2)}(\nu) + \mathcal{O}(\lambda_0^3). \quad (5.13)$$

A similar computation yields,

$$\gamma_0 = 1 + \mathcal{O}(\lambda_0^2),$$

on  $\Xi$ .

Comparing  $|\hat{E}_+ - E_+|$  and  $|\mu_1 - \hat{E}_+|$ , with Theorem 2.8 and Equation (5.13), we can establish the first part of Theorem 2.11, (2.18), and the second part, (2.19). Similarly, if  $N^{-1/6} \ll \lambda_0 \ll 1$ , we can also prove (2.20) by using Theorem 2.8 and (5.13), and in particular,  $\sigma = m^{(2)}(\nu)$ .

We next show that, for any  $\lambda_0 \gg N^{-1/6}$ , (2.20) holds for some  $\sigma \sim 1$ . We notice that, if  $\lambda_0 < \epsilon$  for some sufficiently small constant  $\epsilon > 0$ , independent of  $N$ , we can prove an estimate on the variance of  $X$  similar to (5.13), i.e.,

$$\frac{1}{2}N^{-1}\lambda_0^2 \leq \text{Var}(X) \leq 2N^{-1}\lambda_0^2,$$

which shows that  $\sigma \sim 1$  when  $\lambda_0 < \epsilon$ . When  $\lambda_0 \geq \epsilon$ , it is obvious that

$$1 - (m_{fc}(E_+))^2 = \int \frac{d\nu(v)}{(\lambda_0 v - \vartheta)^2} - \left( \int \frac{d\nu(v)}{\lambda_0 v - \vartheta} \right)^2 > c > 0,$$



for some constant  $c > 0$ , hence  $\text{Var}(X) \sim N^{-1} \sim N^{-1}\lambda_0^2$ . Thus, we can conclude that  $\sigma \sim 1$  in any case. This shows statement *iii* of Theorem 2.11. Since  $X$  is a Gaussian random variable with  $\text{Var}(X) \sim N^{-1}$ , we see that  $\widehat{E}_+ \rightarrow E_+$  as  $N \rightarrow \infty$ .

It remains to prove (2.22). Expanding (5.12) further and solving it for  $\vartheta$ , we find that

$$\vartheta = 1 + \frac{3\lambda_0^2 m^{(2)}(\nu)}{2} + 2\lambda_0^3 m^{(3)}(\nu) + \lambda_0^4 \left( \frac{5m^{(4)}(\nu)}{2} - \frac{45(m^{(2)}(\nu))^2}{4} \right) + O(\lambda_0^5).$$

Thus, we obtain that

$$\begin{aligned} E_+ &= \vartheta - m_{fc}(E_+) = \vartheta - \int \frac{d\nu(v)}{\lambda_0 v - \vartheta} \\ &= \vartheta + \frac{1}{\vartheta} + \frac{\lambda_0^2 m^{(2)}(\nu)}{\vartheta^3} + \frac{\lambda_0^3 m^{(3)}(\nu)}{\vartheta^4} + \frac{\lambda_0^4 m^{(4)}(\nu)}{\vartheta^5} + O(\lambda_0^5) \\ &= 2 + \lambda_0^2 m^{(2)}(\nu) + \lambda_0^3 m^{(3)}(\nu) + \lambda_0^4 \left( m^{(4)}(\nu) - \frac{9(m^{(2)}(\nu))^2}{4} \right) + O(\lambda_0^5), \end{aligned}$$

proving (2.22). This completes the proof of Theorem 2.11.  $\square$

## 6. DYSON BROWNIAN MOTION

In this section, we prove Proposition 5.2. The guiding idea of our proof is that Dyson's Brownian motion (DBM) interpolates (in the sense of distributions) between the deformed Wigner matrix  $H$  and the GOE matrix  $W^{\text{GOE}}$ . Following the flow of the DBM we show that the expectations of the Green functions of  $H$  and  $W^{\text{GOE}}$  can be compared for appropriately chosen energies.

We first recapitulate Dyson's Brownian motion [12] in Subsection 6.1.

**6.1. Preliminaries.** Let  $H_0$  be the matrix

$$H_0 = \lambda_0 V + W, \quad (6.1)$$

where  $V = \text{diag}(v_i)$  is a diagonal matrix and  $W$  is a real symmetric Wigner matrix that satisfies the assumptions in Definition 2.1 and has vanishing diagonal entries (see Subsection 4.1). Here  $\lambda_0$  and  $(v_i)$  are chosen to satisfy Assumption 2.4. We consider  $(v_i)$  to be fixed, in particular, if  $V$  is a random we consider them to be conditioned on the event  $\Xi$  introduced in Definition 4.2.

Let  $(\beta_{ij}(t))$  be a real symmetric matrix, whose diagonal entries are zero and the off-diagonal entries are a collection of independent, up to the symmetry constraint, real standard Brownian motions, independent of  $H_0$ . More precisely,  $\beta_{ii}(t) = 0$ ,  $t \geq 0$ , while  $\beta_{ij}(t)$ ,  $i < j$ ,  $t \geq 0$ , is a standard Brownian motion starting at zero.

Let  $H(t) = (h_{ij}(t))$ ,  $t \geq 0$ , satisfy the stochastic differential equation,

$$dh_{ii} = -\frac{1}{2}h_{ii} dt, \quad dh_{ij} = \frac{d\beta_{ij}}{\sqrt{N}} - \frac{1}{2}h_{ij} dt, \quad (i \neq j), \quad (6.2)$$

with initial condition  $H(t=0) = H_0$ . In the following we usually write  $h_{ij} \equiv h_{ij}(t)$  and we refer to  $t$  as time. Note that we consider in (6.2) a matrix-valued Ornstein-Uhlenbeck process with a drift term which insures that the variances of  $h_{ij}(t)$  remain constant over time. In Dyson's original work [12] this drift term was absent while the diagonal entries were also driven by Brownian motions. It is easy to check that the distribution of  $H(t)$  agrees with the distribution of the matrix

$$\lambda_0 e^{-t/2} V + e^{-t/2} W + (1 - e^{-t})^{1/2} W^{\text{GOE}},$$

where  $W^{\text{GOE}}$  is a GOE matrix independent of  $V$  and  $W$ , whose diagonal entries are set to zero. Thus the process defined by (6.2) indeed interpolates in the sense of distributions between the deformed ensemble and the GOE (with vanishing diagonal) which is invariant under the process defined in (6.2).

In the following we denote by  $\mathbb{E}$  to expectation with respect to the off-diagonal random variables  $(h_{ij})$ ,  $i \neq j$ , while we use the notation  $\mathbb{E}_V$  for the expectation with respect the diagonal random variables  $(h_{ii})$ .

Recall the definition of the  $\lambda$ -dependent quantities  $\zeta \equiv \zeta(\lambda)$  and  $\gamma \equiv \gamma(\lambda)$  in (4.6) and (4.7). Setting  $\lambda(t) := \lambda_0 e^{-t/2}$ , we may now view  $\zeta$ ,  $\gamma$  and  $\lambda$  as depending on  $t$  (and  $\lambda_0$ ) by extending the definitions in (4.6) and (4.7) in the natural way. In the same way, we obtain a  $t$ -dependent measure  $\widehat{\rho}_{fc}(t)$  (whose density at  $E \in \mathbb{R}$  is denote by  $\widehat{\rho}_{fc}(t, E)$ ) by choosing  $\gamma$  and  $\lambda$  depending on  $t$  via  $\lambda \equiv \lambda_0 e^{-t/2}$  in the defining equation (4.9) for  $\widehat{m}_{fc} \equiv \widehat{m}_{fc}(t)$ . Note that the statements of Lemma 4.3 directly carry over to  $\widehat{\rho}_{fc}(t)$  and  $\widehat{m}_{fc}(t)$ . We denote by  $\widehat{L}_+ \equiv \widehat{L}_+(t)$  the upper endpoint of the support of the measure  $\widehat{\rho}_{fc}(t)$ .

We now consider the Green function of the rescaled random matrix  $\tilde{H}(t) := \gamma(t)H(t)$ . To prove Proposition 5.2, we also have to choose the spectral parameter  $z$  as time dependent. Fix some small  $\epsilon > 0$  and define the domain,  $\mathcal{E}_\epsilon \equiv \mathcal{E}_\epsilon(t)$ , of the spectral parameter  $z$  by

$$\mathcal{E}_\epsilon(t) := \{z = L_+(t) + y + i\eta \in \mathbb{C}^+ : y \in [-N^{-2/3+\epsilon}, N^{-2/3+\epsilon}], \eta = N^{-2/3-\epsilon}\}, \quad (t \geq 0). \quad (6.3)$$

For  $z \equiv z(t) \in \mathcal{E}_\epsilon(t)$ , we consider the Green functions

$$G(t, z) := \frac{1}{\gamma(t)H(t) - z(t)}, \quad m(t, z) = \frac{1}{N} \text{Tr} G(t, z), \quad (t \geq 0).$$

Recalling Lemma 4.3 and Theorem 4.6 we obtain that, on  $\Xi$ ,

$$|m(t, z) - \widehat{m}_{fc}(t, z)| \prec \Psi, \quad \max_{i \neq j} |G_{ij}(t, z)| \prec \Psi, \quad \max_i |G_{ii}(t, z)| \prec 1, \quad (6.4)$$

uniformly in  $\mathcal{E}_\epsilon(t)$ ,  $t \geq 0$ , where we have set

$$\Psi := N^{-1/3+C'\epsilon}, \quad (6.5)$$

for some constant  $C'$  independent of  $N$ ,  $\epsilon$  and  $t$ . For simplicity, we abbreviate

$$G \equiv G(t, z), \quad m \equiv m(t, z), \quad \widehat{L}_+ \equiv \widehat{L}_+(t),$$

etc., in the following. Note that, for fixed  $t \geq 0$  and  $\epsilon > 0$ , the spectral parameter  $z$  is a function of  $y \in \mathbb{R}$  (with  $|y| \leq N^{-2/3+\epsilon}$ ) and so are  $G$  and  $m$ .

**6.2. Proof of Proposition 5.2.** In this subsection we give the proof of Proposition 5.2.

Using Itô's lemma we derive the stochastic differential equation for the matrix entries  $G_{ij}(t, z)$  in Subsection 6.3. Anticipating this computation and further calculations of the remaining sections, we next state the key result, Proposition 6.1 below, that directly leads to the proof of Proposition 5.2. Recall that we use the symbol  $\mathbb{E}$  to denote the expectation with respect to the off-diagonal random variables  $(h_{ij})$ ,  $i \neq j$ , while we use the notation  $\mathbb{E}_V$  for the expectation with respect to the diagonal random variables  $(h_{ii})$ .

**Proposition 6.1.** *Let  $H \equiv H(t)$  be the solution of (6.2). Let*

$$\mathfrak{X} := N \int_{E_1}^{E_2} \text{Im} m(\widehat{L}_+ + x - 2 + i\eta) dx = N \int_{E_1-2}^{E_2-2} \text{Im} m(\widehat{L}_+ + \tilde{y} + i\eta) d\tilde{y}, \quad (6.6)$$

where  $E_1, E_2$  satisfy (5.6). Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function satisfying (5.7). Then there exist a real-valued function  $(t, H) \mapsto \Theta(t, H) \equiv \Theta$  and a martingale  $t \mapsto M(t)$  with zero expectation such that

$$dF(\mathfrak{X}) = \Theta dt + dM. \quad (6.7)$$

On  $\Xi$ ,  $\Theta$  satisfies

$$|\mathbb{E}[\Theta(t, H(t))]| \leq CN^{1/2}\Psi^2, \quad (6.8)$$

uniformly in  $t \geq 0$ , where  $\Psi$  is given by (6.5).

Assuming that Proposition 6.1 holds, we can easily prove the desired result, Proposition 5.2.

*Proof of Proposition 5.2.* Since  $|\mathbb{E}\Theta(t, H(t))| \leq CN^{1/2}\Psi^2$ , integrating  $dF(\mathfrak{X})$  from  $t = 0$  to  $t = 4 \log N$  and taking the expectation, we find from Proposition 6.1 that, on  $\Xi$ ,

$$\left| \mathbb{E} \left[ F \left( N \int_{E_1}^{E_2} \text{Im} m(\widehat{L}_+ + x - 2 + i\eta) \Big|_{t=0} dx \right) \right] - \mathbb{E} \left[ F \left( N \int_{E_1}^{E_2} \text{Im} m(\widehat{L}_+ + x - 2 + i\eta) \Big|_{t=4 \log N} dx \right) \right] \right| \leq N^{-1/6+C'\epsilon}, \quad (6.9)$$

for some constant  $C' > 0$ , where we used (6.5).

At  $t = 4 \log N$ , we have  $\lambda = \lambda_0 N^{-2}$ , hence  $\gamma = 1 + O(N^{-2})$  on  $\Xi$ . In particular, the distribution of  $\tilde{H}(t)$  with  $t = 4 \log N$  agrees with the distribution of the matrix

$$\frac{\gamma}{N^2}(\lambda_0 V + W) + \gamma \left( 1 - \frac{1}{N^4} \right)^{1/2} W^{\text{GOE}}. \quad (6.10)$$

Denoting by  $\mu_1^{\text{GOE}} \geq \mu_2^{\text{GOE}} \geq \dots \geq \mu_N^{\text{GOE}}$  the eigenvalues of  $W^{\text{GOE}}$ , and by  $\tilde{\mu}_1 \geq \tilde{\mu}_2 \geq \dots \geq \tilde{\mu}_N$  the eigenvalues of the matrix in (6.10), we have at  $t = 4 \log N$ ,

$$|\tilde{\mu}_j - \mu_j^{\text{GOE}}| = \mathcal{O}_\Xi(N^{-2}).$$

Thus, we have that

$$m(z) = \frac{1}{N} \sum_j \frac{1}{\tilde{\mu}_j - z} = \frac{1}{N} \sum_j \frac{1}{\mu_j^{\text{GOE}} - z} + \mathcal{O}(N^{-4/3+\epsilon}) = m^{\text{GOE}}(z) + \mathcal{O}_{\Xi}(N^{-4/3+\epsilon}),$$

and, since  $\widehat{L}_+ - 2 = O(N^{-2})$  on  $\Xi$ , we have

$$\int_{E_1}^{E_2} \text{Im } m(x + \widehat{L}_+ - 2 + i\eta) \Big|_{t=4 \log N} dx - \int_{E_1}^{E_2} \text{Im } m^{\text{GOE}}(x + i\eta) dx = \mathcal{O}_{\Xi}(N^{-2+C\epsilon}).$$

Hence

$$\left| \mathbb{E} \left[ F \left( N \int_{E_1}^{E_2} \text{Im } m(x + \widehat{L}_+ - 2 + i\eta) \Big|_{t=4 \log N} dx \right) \right] - \mathbb{E} \left[ F \left( N \int_{E_1}^{E_2} \text{Im } m^{\text{GOE}}(x + i\eta) dx \right) \right] \right| \leq N^{-1+C\epsilon}.$$

Using the uniform boundedness of  $F$ , we obtain

$$\begin{aligned} & \left| \mathbb{E}_V \mathbb{E} \left[ F \left( N \int_{E_1}^{E_2} \text{Im } m(\widehat{L}_+ + x - 2 + i\eta) \Big|_{t=0} dx \right) \right] - \mathbb{E}_V \mathbb{E} \left[ F \left( N \int_{E_1}^{E_2} \text{Im } m^{\text{GOE}}(x + i\eta) dx \right) \right] \right| \\ & \leq CN^{-1/6+C'\epsilon} + CN^{-\epsilon'\alpha}, \end{aligned}$$

where we used that  $\mathbb{P}_V(\Xi) \leq CN^{-\epsilon'\alpha}$  with  $\alpha = \min\{\beta_0, \beta_1\}$ , by Assumption 2.2 and Assumption 2.4. This completes the proof of Proposition 5.2  $\square$

**6.3. Green function flow.** Recall that we let  $dF(\mathfrak{X}) = \Theta dt + dM$ . To prove Proposition 6.1, we first describe  $\Theta(t, H(t))$ , where  $H(t)$  is the solution to (6.2), in terms of the entries of  $G(t, z)$ .

**Lemma 6.2.** *Under the assumptions of Proposition 6.1 we have*

$$\begin{aligned} & \mathbb{E}[\Theta(t, H(t))] \\ &= \sum_{i,a} \left( -\partial_t(\lambda\gamma)v_a \mathbb{E} \left[ F'(\mathfrak{X}) \left( \text{Im} \int_{E_1-2}^{E_2-2} G_{ia} G_{ai} dy \right) \right] + \dot{z} \mathbb{E} \left[ F'(\mathfrak{X}) \left( \text{Im} \int_{E_1-2}^{E_2-2} G_{ia} G_{ai} dy \right) \right] \right) \\ &+ \frac{2\dot{\gamma}\gamma}{N} \sum_i \sum_{a \neq b} \left( \mathbb{E} \left[ F'(\mathfrak{X}) \left( \text{Im} \int_{E_1-2}^{E_2-2} G_{ia} G_{ab} G_{bi} dy \right) \right] + \mathbb{E} \left[ F'(\mathfrak{X}) \left( \text{Im} \int_{E_1-2}^{E_2-2} G_{ia} G_{bb} G_{ai} dy \right) \right] \right) \\ &+ \frac{2\dot{\gamma}\gamma}{N} \sum_{i,j} \sum_{a \neq b} \mathbb{E} \left[ F''(\mathfrak{X}) \left( \text{Im} \int_{E_1-2}^{E_2-2} G_{ja} G_{bj} dy \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} G_{ia} G_{bi} dy \right) \right] + \mathcal{O}_{\Xi}(N^{1/2}\Psi^2), \end{aligned} \quad (6.11)$$

where we abbreviate  $G \equiv G(\widehat{L}_+ + y + i\eta)$ . Moreover, recalling that  $z \equiv z(t) = \widehat{L}_+(t) + y + i\eta \in \mathcal{E}_{\epsilon}(t)$ , we have

$$\dot{z}(t) = -2\gamma\dot{\gamma}\widehat{m}_{fc}(\widehat{L}_+) + \gamma^2 \partial_t(\lambda\gamma) \frac{1}{N} \sum_{i=1}^N \frac{v_i}{(\lambda\gamma v_i - \tau)^2}, \quad (6.12)$$

for  $t \geq 0$  on  $\Xi$ .

We prove Lemma 6.2 in Subsection 6.4. To illustrate the essence of the proof, we first consider the differentials  $dG_{ij}$  of the Green function  $G(t, z)$ . Using Itô's lemma, we compute

$$\begin{aligned} dG_{ij} &= \frac{\partial G_{ij}}{\partial t} dt + \sum_{a \leq b} \frac{\partial G_{ij}}{\partial h_{ab}} dh_{ab} + \sum_{a \leq b} \sum_{c \leq d} \frac{1}{2} \frac{\partial^2 G_{ij}}{\partial h_{ab} \partial h_{cd}} d\langle h_{ab}, h_{cd} \rangle_t \\ &= \frac{\partial G_{ij}}{\partial t} dt + \sum_{a \leq b} \frac{\partial G_{ij}}{\partial h_{ab}} \left( \frac{d\beta_{ab}}{\sqrt{N}} - \frac{h_{ab}}{2} dt \right) - \frac{1}{2} \sum_a \frac{\partial G_{ij}}{\partial h_{aa}} h_{aa} dt \\ &\quad + \sum_{a \leq b} \sum_{c \leq d} \frac{1}{2} \frac{\partial^2 G_{ij}}{\partial h_{ab} \partial h_{cd}} d\langle h_{ab}, h_{cd} \rangle_t. \end{aligned} \quad (6.13)$$

The cross-variance in (6.13) is explicitly given by

$$\langle h_{ab}, h_{ab} \rangle_t = \langle h_{ab}, h_{ba} \rangle_t = \begin{cases} \frac{1}{N} & \text{if } a \neq b, \\ 0 & \text{if } a = b, \end{cases} \quad (6.14)$$

with  $\langle h_{ab}, h_{cd} \rangle_t = 0$  if  $\{a, b\} \neq \{c, d\}$ . Using (6.14) and the symmetries  $h_{ab} = h_{ba}$ , we obtain from (6.13) that

$$dG_{ij} = \left( \frac{\partial G_{ij}}{\partial t} - \frac{1}{2} \sum_a \frac{\partial G_{ij}}{\partial h_{aa}} h_{aa} - \frac{1}{4} \sum_{a \neq b} \frac{\partial G_{ij}}{\partial h_{ab}} h_{ab} + \frac{1}{4N} \sum_{a \neq b} \frac{\partial G_{ij}^2}{\partial h_{ab}^2} \right) dt + dM_{ij}, \quad (6.15)$$

where we have introduced the martingale term

$$dM_{ij} := \sum_{a < b} \frac{\partial G_{ij}}{\partial h_{ab}} \frac{d\beta_{ab}}{\sqrt{N}}. \quad (6.16)$$

Next, we compute the derivatives in (6.15). For the time derivative we obtain

$$\frac{\partial G_{ij}}{\partial t} = \sum_a (-\dot{\gamma} G_{ia} \lambda v_a G_{aj} + \dot{z} G_{ia} G_{aj}) - \sum_{a \neq b} (\dot{\gamma} G_{ia} w_{ab} G_{bj}).$$

For the first spatial derivate we obtain, for  $a \neq b$ ,

$$-\frac{\partial G_{ij}}{\partial h_{aa}} h_{aa} = \lambda \gamma v_a G_{ia} G_{aj}, \quad -\frac{\partial G_{ij}}{\partial h_{ab}} h_{ab} = 2\gamma G_{ia} w_{ab} G_{bj}.$$

For the second spatial derivatives we find, for  $a \neq b$ ,

$$\frac{\partial^2 G_{ii}}{\partial h_{ab}^2} = 2\gamma^2 (G_{ia} G_{ab} G_{bj} + G_{ib} G_{ba} G_{aj} + G_{ib} G_{aa} G_{bj} + G_{ia} G_{bb} G_{aj}).$$

Thus, using  $\partial_t(\lambda\gamma) = -\lambda\gamma/2 + \dot{\gamma}\lambda$ , we can rewrite (6.15) as

$$\begin{aligned} dG_{ij} = & \sum_a \left( -\partial_t(\gamma\lambda) v_a G_{ia} G_{aj} + \dot{z} G_{ia} G_{aj} \right) dt + dM_{ij} \\ & + \sum_{a \neq b} \left( -\dot{\gamma} G_{ia} w_{ab} G_{bj} + \frac{\gamma}{2} G_{ia} w_{ab} G_{bj} + \frac{\gamma^2}{N} G_{ia} G_{ab} G_{bj} + \frac{\gamma^2}{N} G_{ia} G_{bb} G_{aj} \right) dt. \end{aligned} \quad (6.17)$$

**Example 6.3.** In the simple case where  $\lambda = \lambda_0 = 0$  and  $\gamma = \gamma_0 = 1$ , we have  $\dot{z} = \partial_t \hat{L}_+ = 0$  and Equation (6.17) becomes

$$dG_{ij} = \sum_{a,b} \left( \frac{1}{2} G_{ia} w_{ab} G_{bj} + \frac{1}{N} G_{ia} G_{ab} G_{bj} + \frac{1}{N} G_{ia} G_{bb} G_{aj} \right) dt + dM_{ij}. \quad (6.18)$$

Note that in this simple example  $H(t=0)$  reduces to a real symmetric Wigner matrix (with vanishing diagonal) and we have  $\hat{m}_{fc} \equiv m_{sc}$ , where  $m_{sc}$  is the Stieltjes transform of the standard semicircle law  $\rho_{sc}$ .

Eventually, we are going to take the expectation of (6.18). To compute the expectation of  $G_{ia} w_{ab} G_{bj}$ , we use the following lemma that was used in the context of random matrix theory before in [23], see also [37, 36]. For a function  $Q$  of the matrix entry  $h_{ab}$ , we denote  $\partial_{ab}^m Q \equiv \frac{\partial^m}{\partial h_{ab}^m} Q$ ,  $m \in \mathbb{N}$ .

**Lemma 6.4.** Assume that  $Q \in C^{M+1}(\mathbb{R})$  for some  $M \in \mathbb{N}$ . Then,

$$\mathbb{E} \partial_{ab} Q(h_{ab}) h_{ab} = \sum_{m=1}^M \frac{\kappa_{ab}^{(m)}}{(m-1)!} \mathbb{E} [\partial_{ab}^m Q(h_{ab})] + O(\| \partial_{ab}^{M+1} Q \|_\infty \kappa_{ab}^{(M+1)}), \quad (6.19)$$

where  $(\kappa_{ab}^{(m)})$ ,  $m \in \mathbb{N}$ , are the cumulants of  $(h_{ab})$ .

*Proof.* By assumption it suffices to check (6.19) for monomials up to order  $M$ . For monomials (6.19) is a direct consequence of the moment-cumulant relation

$$\kappa_{ab}^{(n)} = M_{ab}^{(n)} - \sum_{m=1}^{n-1} \binom{n-1}{m-1} \kappa_{ab}^{(n)} M_{ab}^{(n-m)},$$

where  $(M_{ab}^{(n)})$ ,  $(\kappa_{ab}^{(n)})$  are the moments, respectively cumulants of  $(h_{ab})$ . □

Note that we have, by Assumption 2.1 and the definition of the cumulants, for  $a \neq b$ ,

$$\kappa_{ab}^{(1)} = 0, \quad \kappa_{ab}^{(2)} = \frac{1}{N}, \quad \kappa_{ab}^{(p)} \leq \frac{k_p}{N^{p/2}}, \quad (k \geq 3),$$

for constants  $(k_p)$  independent of  $N$ .

Choosing  $Q \equiv G_{ij}$ , we get from Lemma 6.4, for  $a \neq b$ ,

$$\mathbb{E} \frac{\partial G_{ij}}{\partial h_{ab}} h_{ab} = \frac{1}{N} \mathbb{E} \frac{\partial^2 G_{ij}}{\partial h_{ab}^2} + \frac{1}{2} \kappa_{ab}^{(3)} \mathbb{E} \frac{\partial^3 G_{ij}}{\partial h_{ab}^3} + \mathcal{O}(N^{-2} \Psi^2). \quad (6.20)$$

The first term on the right side of (6.20) can be handled with the following lemma whose proof appeared first in [19].

**Lemma 6.5.** *For  $i \neq a \neq b \neq j$ , we have*

$$\left| \mathbb{E} \frac{\partial^3 G_{ij}}{\partial h_{ab}^3} \right| \prec \Psi^3. \quad (6.21)$$

*Proof.* To show (6.21) it clearly suffices to control  $\frac{\partial^2}{\partial w_{ab}^2}(G_{ia}G_{bj})$ . Observe that each term in this last expression contains at least three off-diagonal resolvent entries except  $G_{ia}G_{aa}G_{bb}G_{bj}$  and  $G_{ib}G_{aa}G_{bb}G_{aj}$ . We will focus on the former term, the latter can be treated in the very same way. Using  $|G_{aa} - m_{sc}| \prec \Psi$  (see Theorem 4.6, with  $\lambda = 0$ ,  $\gamma = 1$ ) we get

$$\mathbb{E} G_{ia}G_{aa}G_{bb}G_{bj} = m_{sc}^2 \mathbb{E} G_{ia}G_{bj} + \mathcal{O}(\Psi^3). \quad (6.22)$$

Using the resolvent identity (3.10), we may write

$$G_{ia} = -G_{aa}G_{ii}^{(a)} \left( h_{ia} - \sum_{p,q}^{(a)} h_{ip} G_{pq}^{(a)} h_{qa} \right) = m_{sc}^2 \left( h_{ia} - \sum_{p,q}^{(a)} h_{ip} G_{pq}^{(a)} h_{qa} \right) + \mathcal{O}(\Psi^2),$$

where we used once more the local law (4.23). Since the first term on the very right side has vanishing expectation, we obtain from (6.22) that  $|\mathbb{E} G_{ia}G_{aa}G_{bb}G_{bj}| \prec \Psi^3$  which implies the claim.  $\square$

Returning to (6.20), we obtain, for  $i \neq a \neq b \neq j$ ,

$$\begin{aligned} \mathbb{E} \frac{\partial G_{ij}}{\partial w_{ab}} w_{ab} &= -\frac{1}{N} \mathbb{E} G_{ia}G_{ab}G_{bj} - \frac{1}{N} \mathbb{E} G_{ib}G_{ab}G_{aj} - \frac{1}{N} \mathbb{E} G_{ib}G_{aa}G_{bj} \\ &\quad - \frac{1}{N} \mathbb{E} G_{ia}G_{bb}G_{aj} + \mathcal{O}(N^{-3/2} \Psi^3). \end{aligned}$$

In sum, we have shown that

$$\sum_{a \neq b} \mathbb{E} G_{ia}w_{ab}G_{bj} = -\frac{2}{N} \sum_{a \neq b} (\mathbb{E} G_{ia}G_{ab}G_{bj} + \mathbb{E} G_{ib}G_{aa}G_{bj}) + \mathcal{O}(N^{1/2} \Psi^3),$$

where we used  $|G_{ii}w_{ib}G_{bj}|, |G_{ia}w_{aj}G_{jj}|, |\mathbb{E}_i G_{ii}w_{ij}G_{jj}| \prec N^{-1/2} \Psi$ , to cope with the cases  $a = i$ ,  $a \neq b$ , etc.. This shows that (6.18) can be written as

$$\mathbb{E} dG_{ij}(z, t) = \mathcal{O}(N^{1/2} \Psi^3) dt + \mathcal{O}(\Psi^2) dt + \mathcal{O}(N^{-1/2} \Psi) dt = \mathcal{O}(N^{1/2} \Psi^3) dt,$$

(uniformly in  $t \geq 0$ ), where we used that the expectation of the martingale term defined in (6.16) vanishes. Integration over  $t$  from 0 to  $4 \log N$ , leads to

$$\mathbb{E} \left( \frac{1}{W - z} \right)_{ij} - \mathbb{E} \left( \frac{1}{W^{\text{GOE}} - z} \right)_{ij} = \mathcal{O}(N^{1/2} \Psi^3),$$

which is stronger an estimate than the trivial bound  $\mathcal{O}(\Psi)$  obtained from the local laws in Theorem 4.6.

**6.4. Computation of  $dF(\mathfrak{X})$  and proof of Lemma 6.2.** We now turn to the computation of the differential  $dF(\mathfrak{X})$ , where  $F$  is a smooth function satisfying (5.7) and where  $\mathfrak{X}$  is defined in (6.6). Choosing  $i = j$  in (6.17), we get

$$\begin{aligned} dG_{ii} &= \sum_a \left( -\partial_t(\gamma\lambda) v_a G_{ia}G_{ai} + \dot{z} G_{ia}G_{ai} \right) dt + dM_{ii} \\ &\quad + \sum_{a \neq b} \left( -\dot{\gamma} G_{ia}w_{ab}G_{bi} + \frac{\gamma}{2} G_{ia}w_{ab}G_{bi} + \frac{\gamma^2}{N} G_{ia}G_{ab}G_{bi} + \frac{\gamma^2}{N} G_{ia}G_{bb}G_{ai} \right) dt. \end{aligned} \quad (6.23)$$

with the martingale term

$$dM_{ii} = \sum_{a < b} \frac{\partial G_{ii}}{\partial h_{ab}} \frac{d\beta_{ab}}{\sqrt{N}} = \frac{\gamma}{\sqrt{N}} \sum_{a \neq b} G_{ia}G_{bi} d\beta_{ab}.$$

Recalling the definitions of  $\mathfrak{X}$  in Proposition 6.1, we obtain from Itô's lemma and (6.23),

$$\begin{aligned} dF(\mathfrak{X}) &= F'(\mathfrak{X}) \sum_{i,a} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} dy \left( -\partial_t(\gamma\lambda) v_a G_{ia} G_{ai} + \dot{z} G_{ia} G_{ai} \right) \right) dt \\ &\quad + F'(\mathfrak{X}) \sum_i \sum_{a \neq b} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} dy \left( \left( \frac{\gamma}{2} - \dot{\gamma} \right) G_{ia} w_{ab} G_{bi} + \frac{\gamma^2}{N} G_{ia} G_{ab} G_{bi} + \frac{\gamma^2}{N} G_{ia} G_{bb} G_{ai} \right) \right) dt \\ &\quad + F''(\mathfrak{X}) \frac{\gamma^2}{N} \sum_{i,j} \sum_{a \neq b} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} dy G_{ia} G_{bi} \right) \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} dy G_{ja} G_{bj} \right) dt + dM, \end{aligned} \quad (6.24)$$

for some martingale  $M$  of vanishing expectation. Here, we use the notation  $G \equiv G(y + L_+ + i\eta)$ . We remark that (6.24) gives rise to the definitions of  $\Theta$  and  $M$  in  $dF(\mathfrak{X}) = \Theta dt + dM$  in Proposition 6.1.

Next, we take the expectation in (6.24). The resulting expression can be treated following the lines of Example 6.3: For  $a \neq b$ , set

$$R(w_{ab}) := F'(\mathfrak{X}) G_{ia} G_{bi}. \quad (6.25)$$

The following lemma bounds  $R$ .

**Lemma 6.6.** *Let  $R(w_{ab}) := F'(\mathfrak{X}) G_{ia} G_{bi}$ . Then, for  $i \neq a \neq b \neq i$ , we have*

$$\mathbb{E} \frac{\partial^2 R}{\partial w_{ab}^2} = \mathcal{O}_\Xi(\Psi^3). \quad (6.26)$$

Here  $\Xi$  denotes the event defined in Definition 4.2.

Lemma 6.6 is proven in the same way as Lemma 6.5, but its proof is lengthier due to more notation and is therefore postponed to the Appendix D.

From Lemma 6.6, with  $i \neq a \neq b \neq i$ , we obtain

$$\begin{aligned} \mathbb{E}[R(w_{ab}) w_{ab}] &= -\frac{\gamma}{N} \left( 2\mathbb{E}[F'(\mathfrak{X}) G_{ia} G_{ab} G_{bi}] + \mathbb{E}[F'(\mathfrak{X}) G_{ib} G_{aa} G_{bi}] + \mathbb{E}[F'(\mathfrak{X}) G_{ia} G_{bb} G_{ai}] \right) \\ &\quad - 2\frac{\gamma}{N} \mathbb{E} \left[ F''(\mathfrak{X}) \sum_j \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} d\tilde{y} \tilde{G}_{ja} \tilde{G}_{bj} \right) G_{ia} G_{bi} \right] + \mathcal{O}_\Xi(N^{-3/2} \Psi^3), \end{aligned}$$

where we use the notation  $\tilde{G} \equiv G(\hat{L}_+ + \tilde{y} + i\eta)$ , respectively  $G \equiv G(\hat{L}_+ + y + i\eta)$ . Altogether, we have that

$$\begin{aligned} \sum_{a \neq b} \mathbb{E}[F'(\mathfrak{X}) G_{ia} w_{ab} G_{bi}] &= -\frac{2\gamma}{N} \sum_{a \neq b} \left( \mathbb{E}[F'(\mathfrak{X}) G_{ia} G_{ab} G_{bi}] + \mathbb{E}[F'(\mathfrak{X}) G_{ib} G_{aa} G_{bi}] \right) \\ &\quad - \frac{2\gamma}{N} \sum_j \sum_{a \neq b} \mathbb{E} \left[ F''(\mathfrak{X}) \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} d\tilde{y} \tilde{G}_{ja} \tilde{G}_{bj} \right) G_{ia} G_{bi} \right] + \mathcal{O}_\Xi(N^{1/2} \Psi^3), \end{aligned} \quad (6.27)$$

uniformly in  $t \geq 0$ . Next we prove Lemma 6.2.

*Proof of Lemma 6.2.* Combining (6.27) with (6.24), we obtain (6.11). We remark that, after integrating over the interval  $[E_1 - 2, E_2 - 2]$  and summing over the index  $i$ , we get an additional factor  $N\Psi^2$ . Thus, the error term of order  $\mathcal{O}_\Xi(N^{1/2}\Psi^3)$  in (6.27) becomes  $\mathcal{O}_\Xi(N^{3/2}\Psi^5)$ , which is equivalent to  $\mathcal{O}_\Xi(N^{1/2}\Psi^2)$ .

It remains to prove (6.12). Recall the definition of  $\tau$  in (4.19). Using that  $\hat{L}_+ = \tau - \gamma^2 \hat{m}_{fc}(\hat{L}_+)$ , we compute

$$\begin{aligned} \partial_t \hat{L}_+ &= \partial_t \tau - 2\gamma \dot{\gamma} \hat{m}_{fc}(\hat{L}_+) - \gamma^2 \partial_t \hat{m}_{fc}(\hat{L}_+) \\ &= \partial_t \tau - 2\gamma \dot{\gamma} \hat{m}_{fc}(\hat{L}_+) + \gamma^2 \frac{1}{N} \sum_{j=1}^N \frac{1}{(\lambda \gamma v_j - \tau)^2} (\partial_t(\lambda \gamma) v_j - \partial_t \tau). \end{aligned}$$

From (4.18), we find

$$\frac{1}{N} \sum_{j=1}^N \frac{1}{(\lambda \gamma v_j - \tau)^2} = \frac{1}{\gamma^2},$$



which, after differentiation with respect to  $t$ , yields

$$\begin{aligned}\partial_t \widehat{L}_+ &= \partial_t \tau - 2\gamma \dot{\gamma} \widehat{m}_{fc}(\widehat{L}_+) + \gamma^2 \partial_t(\lambda \gamma) \frac{1}{N} \sum_{j=1}^N \frac{v_j}{(\lambda \gamma v_j - \tau)^2} - \partial_t \tau \\ &= -2\gamma \dot{\gamma} \widehat{m}_{fc}(\widehat{L}_+) + \gamma^2 \partial_t(\lambda \gamma) \frac{1}{N} \sum_{j=1}^N \frac{v_j}{(\lambda \gamma v_j - \tau)^2}.\end{aligned}$$

This shows (6.12) and thus completes the proof of Lemma 6.2.  $\square$

To conclude this section, we return to Example 6.3, where we considered  $\lambda = \lambda_0 = 0$  and  $\gamma = \gamma_0 = 1$ . Setting  $\lambda = \dot{\gamma} = \dot{z} = 0$  in Equation (6.11), we find that

$$\mathbb{E}[\Theta(t, H(t))] = \mathcal{O}(N^{1/2} \Psi^2),$$

for all  $t \geq 0$ . Integration of  $dF(\mathfrak{X}) = \mathcal{O}(N^{1/2} \Psi^2) dt$  from  $t = 0$  to  $t = 4 \log N$  as in the proof of Proposition 6.2 yields now a simple proof of the fact that the distribution of the largest eigenvalue of Wigner matrix is given by the Tracy-Widom distribution.

If, however,  $\lambda_0 \neq 0$ ,  $\gamma_0 \neq 1$ , and thus  $\dot{\gamma} \neq 0$ ,  $\dot{z} \neq 0$ , the leading terms on the right side of (6.11) are a priori of order one. In the remaining sections, we are going to show that these terms cancel for our choices of  $\gamma$  and  $\dot{z}$  up to errors of order  $N^{1/2} \Psi^2$ . Since this cancellation mechanism in the Green function flow involves rather subtle computations, we first present the main ideas in Section 7 for the simple case  $\lambda_0 \ll 1$ .

## 7. GREEN FUNCTION FLOW - A SIMPLE CASE

In this section, we assume that  $\lambda_0 = N^{-\delta}$  for some  $\delta > 0$ , i.e., we consider  $H = N^{-\delta} V + W$ . For such small  $\lambda$ , we get from (4.7) that  $\gamma_0 = 1 + O(\lambda)$ . As shown below, we may set  $\gamma_0 \equiv \gamma \equiv 1$  for simplicity of the exposition since the error term of order  $\lambda$  is negligible. Furthermore, we let  $F' \equiv 1$  so that the conclusion of Proposition 6.1 becomes

$$\mathbb{E}[\Theta] = \frac{1}{N} \sum_{i,a} \text{Im} \int_{E_1-2}^{E_2-2} \left( -\dot{\lambda} v_a \mathbb{E}[G_{ia} G_{ai}] + \dot{z} \mathbb{E}[G_{ia} G_{ai}] \right) dy + \mathcal{O}_{\Xi}(N^{1/2} \Psi^3). \quad (7.1)$$

In this section we prove that

$$\text{Im} \sum_a \left( -\dot{\lambda} v_a \mathbb{E}[G_{ia} G_{ai}] + \dot{z} \mathbb{E}[G_{ia} G_{ai}] \right) = \mathcal{O}_{\Xi}(N^{-\delta} \Psi), \quad (7.2)$$

which also implies that  $\mathbb{E}[\Theta] = \mathcal{O}_{\Xi}(N^{-\delta+C\epsilon})$ . We remark that the bound in (7.2) is non-trivial in the sense that the naive power-counting from the local law only yields a bound of  $\mathcal{O}_{\Xi}(\Psi)$ .

The main difficulty to overcome in the proof of (7.2) is that the index  $a$  appears in the deterministic part  $v_a$  as well as the random part  $G_{ia} G_{ai}$  in the first term. If we can “decouple” the index  $a$  from the resolvent entries in the sense that we can choose a (non-random) function  $f$  such that

$$\sum_a \dot{\lambda} v_a \mathbb{E}[G_{ia} G_{ai}] = \left( \frac{1}{N} \sum_a f(v_a) \right) \sum_{s=1}^N \mathbb{E}[G_{is} G_{si}] + \mathcal{O}_{\Xi}(N^{-\delta} \Psi),$$

we can prove (7.2) by comparing the coefficient  $N^{-1} \sum_a f(v_a)$  with  $\dot{z}$  in the second term in (7.2). In Subsection 7.1 we illustrate the ideas behind this “decoupling mechanism” for the index  $a$ .

**7.1. Expansion of  $\mathbb{E}[G_{ia} G_{ai}]$ .** We introduce a procedure that renders the indices of the random part  $G_{ia} G_{ai}$  free of the index  $a$ . We proceed in three steps:

*Step 1.* In a first step, we remove the index  $a$  in the lower indices of the resolvents. We begin by using the resolvent formulas in (3.9) that read

$$G_{ia} = -G_{aa} \sum_s^{(a)} G_{is}^{(a)} h_{sa}, \quad G_{ai} = -G_{aa} \sum_t^{(a)} h_{at} G_{ti}^{(a)},$$

where we assume  $a \neq i$  at first. Later, we are going to add the term  $a = i$ , which in fact is negligible for the case at hand ( $\lambda_0 \ll 1, \gamma = 1$ ). We then have

$$G_{ia} G_{ai} = G_{aa}^2 \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)}.$$

Using Schur's complement formula (3.7), we rewrite this as

$$G_{ia}G_{ai} = \frac{1}{(\lambda v_a - z - \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa})^2} \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)}.$$

Let  $\tau$  be the largest solution to the equation

$$\frac{1}{N} \sum_j \frac{1}{(\lambda v_j - \tau)^2} = 1.$$

Then, from the large deviation estimates in Lemma 3.3 and the local laws in (6.4), we have

$$z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau = \mathcal{O}_\Xi(\Psi), \quad z + m^2 - \tau = \mathcal{O}_\Xi(\Psi).$$

Thus, we get

$$\begin{aligned} G_{ia}G_{ai} &= \frac{1}{(\lambda v_a - \tau)^2} \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} + \frac{2}{(\lambda v_a - \tau)^3} \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \\ &\quad + \mathcal{O}_\Xi(\Psi^4). \end{aligned} \quad (7.3)$$

Note that the index  $a$  does not appear as an lower index of the resolvent terms on the right side, yet every resolvent term has  $a$  in the upper index. We remark that the terms of  $\mathcal{O}_\Xi(\Psi^4)$  is negligible for  $\lambda_0 \ll 1$ .

*Step 2.* In a second step, we integrate out the matrix entries labeled by the index  $a$  (i.e.,  $h_{ax}$  and  $h_{ya}$ , for some  $x$  and  $y$ ) on the right side of (7.3) by taking the partial expectation  $\mathbb{E}_a$  with respect to the  $a$ -th column and row. More precisely, we consider

$$\begin{aligned} \mathbb{E}[G_{ia}G_{ai}] &= \mathbb{E}[\mathbb{E}_a[G_{ia}G_{ai}]] \\ &= \mathbb{E} \left[ \frac{1}{(\lambda v_a - \tau)^2} \mathbb{E}_a \left[ \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \right] \right] \\ &\quad + \mathbb{E} \left[ \frac{2}{(\lambda v_a - \tau)^3} \mathbb{E}_a \left[ \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \right] \right] + \mathcal{O}_\Xi(\Psi^4). \end{aligned}$$

In the first term we have

$$\mathbb{E}_a \left[ \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \right] = \frac{1}{N} \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)},$$

and, similarly in the second term, we have

$$\begin{aligned} &\mathbb{E}_a \left[ \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \right] \\ &= (z + m^{(a)} - \tau) \frac{1}{N} \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} + \frac{2}{N^2} \sum_{p,q,s}^{(a)} G_{ip}^{(a)} G_{pq}^{(a)} G_{qi}^{(a)} + \mathcal{O}_\Xi(\Psi^4), \end{aligned}$$

where the first term comes from the case  $p = q$  and  $s = t$ , while the second term from  $p = s$  and  $q = t$  or  $p = t$  and  $q = s$ . Here we also used the fact that  $G$  is symmetric and that the contribution from the case  $p = q = s = t$  is negligible. We thus have

$$\begin{aligned} \mathbb{E}[G_{ia}G_{ai}] &= \frac{1}{(\lambda v_a - \tau)^2} \mathbb{E} \left[ \frac{1}{N} \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} \right] + \frac{2}{(\lambda v_a - \tau)^3} \mathbb{E} \left[ (z + m^{(a)} - \tau) \frac{1}{N} \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} \right] \\ &\quad + \frac{4}{(\lambda v_a - \tau)^3} \mathbb{E} \left[ \frac{1}{N^2} \sum_{p,q,s}^{(a)} G_{ip}^{(a)} G_{pq}^{(a)} G_{qi}^{(a)} \right] + \mathcal{O}_\Xi(\Psi^4). \end{aligned} \quad (7.4)$$

Note that, at the end of this second step, only resolvent terms remain. Also note that the index  $a$  appears now as an upper index.

*Step 3.* In a third step, we remove the upper index  $a$  in the resolvent entries by using the formula (3.8) that reads

$$G_{is}^{(a)} = G_{is} - \frac{G_{ia}G_{as}}{G_{aa}}.$$

Recall that  $G_{ia}G_{as} = \mathcal{O}_\Xi(\Psi^2)$ , if  $a \neq i, s$ .

We thus have for the first term in (7.4) that

$$\mathbb{E} \left[ \frac{1}{N} \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} \right] = \frac{1}{N} \sum_s^{(a)} \mathbb{E} \left[ G_{is} G_{si} - \frac{G_{ia}G_{as}}{G_{aa}} G_{si}^{(a)} - G_{is} \frac{G_{sa}G_{ai}}{G_{aa}} \right].$$

In the first term of the right side of the last equation, we notice that

$$\frac{1}{N} \sum_s^{(a)} G_{is} G_{si} = \frac{1}{N} \sum_s G_{is} G_{si} + \mathcal{O}_\Xi(\Psi^5).$$

Thus, we arrive at

$$\begin{aligned} & \mathbb{E}[G_{ia}G_{ai}] \\ &= \frac{1}{(\lambda v_a - \tau)^2} \mathbb{E} \left[ \frac{1}{N} \sum_s G_{is} G_{si} \right] - \frac{1}{(\lambda v_a - \tau)^2} \mathbb{E} \left[ \frac{1}{N} \sum_s \frac{G_{ia}G_{as}}{G_{aa}} G_{si}^{(a)} \right] \\ & \quad - \frac{1}{(\lambda v_a - \tau)^2} \mathbb{E} \left[ \frac{1}{N} \sum_s G_{is} \frac{G_{sa}G_{ai}}{G_{aa}} \right] + \frac{2}{(\lambda v_a - \tau)^3} \mathbb{E} \left[ (z + m^{(a)} - \tau) \frac{1}{N} \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} \right] \\ & \quad + \frac{4}{(\lambda v_a - \tau)^3} \mathbb{E} \left[ \frac{1}{N^2} \sum_{p,q,s}^{(a)} G_{ip}^{(a)} G_{pq}^{(a)} G_{qi}^{(a)} \right] + \mathcal{O}_\Xi(\Psi^4). \end{aligned} \quad (7.5)$$

Note that the first term on the right side of (7.5) neither has  $a$  as a lower nor an upper index.

After following *Steps 1-3*, we obtain a term we desire: the first term on the right side of (7.5). When a term contains  $a$  neither in the lower index nor in the upper index, as in the first term of (7.5), we call it fully expanded. For the other terms in (7.5), we repeat *Steps 1-3* until every non-negligible term is fully expanded.

For example, we apply *Step 1* to the second term in (7.5) to get

$$\begin{aligned} \frac{1}{N} \sum_s^{(a)} \frac{G_{ia}G_{as}}{G_{aa}} G_{si}^{(a)} &= \frac{1}{N} \sum_s \frac{1}{G_{aa}} G_{aa}^2 \sum_{k,m}^{(a)} G_{ik}^{(a)} h_{ka} h_{am} G_{ms}^{(a)} G_{si}^{(a)} \\ &= \frac{1}{\lambda v_a - \tau} \frac{1}{N} \sum_{s,k,m}^{(a)} G_{ik}^{(a)} h_{ka} h_{am} G_{ms}^{(a)} G_{si}^{(a)} + \mathcal{O}_\Xi(\Psi^4). \end{aligned}$$

By taking the partial expectation, i.e., from *Step 2*, we find that

$$\mathbb{E} \left[ \frac{1}{N} \sum_s^{(a)} \frac{G_{ia}G_{as}}{G_{aa}} G_{si}^{(a)} \right] = \frac{1}{\lambda v_a - \tau} \mathbb{E} \left[ \frac{1}{N^2} \sum_{s,k}^{(a)} G_{ik}^{(a)} G_{ks}^{(a)} G_{si}^{(a)} \right] + \mathcal{O}_\Xi(\Psi^4).$$

Since  $G_{is} - G_{is}^{(a)} = \mathcal{O}_\Xi(\Psi^2)$ , we find after performing *Step 3* that

$$\begin{aligned} \frac{1}{(\lambda v_a - \tau)^2} \mathbb{E} \left[ \frac{1}{N} \sum_s^{(a)} \frac{G_{ia}G_{as}}{G_{aa}} G_{si}^{(a)} \right] &= \frac{1}{(\lambda v_a - \tau)^3} \mathbb{E} \left[ \frac{1}{N^2} \sum_{s,k}^{(a)} G_{ik}^{(a)} G_{ks}^{(a)} G_{si}^{(a)} \right] + \mathcal{O}_\Xi(\Psi^4) \\ &= \frac{1}{(\lambda v_a - \tau)^3} \mathbb{E} \left[ \frac{1}{N^2} \sum_{s,k} G_{ik} G_{ks} G_{si} \right] + \mathcal{O}_\Xi(\Psi^4). \end{aligned}$$

We go through the same procedure for the third term in (7.5). Since it contains  $G_{is}$ , which does not have  $a$  in the indices, we begin by

$$\frac{1}{N} \sum_s^{(a)} G_{is} \frac{G_{sa}G_{ai}}{G_{aa}} = \frac{1}{N} \sum_s G_{is}^{(a)} \frac{G_{sa}G_{ai}}{G_{aa}} + \mathcal{O}_\Xi(\Psi^4).$$

Then, after following *Steps 1-3* again, we obtain that

$$\frac{1}{(\lambda v_a - \tau)^2} \mathbb{E} \left[ \frac{1}{N} \sum_s^{(a)} G_{is} \frac{G_{sa} G_{ai}}{G_{aa}} \right] = \frac{1}{(\lambda v_a - \tau)^3} \mathbb{E} \left[ \frac{1}{N^2} \sum_{s,k} G_{ik} G_{ks} G_{si} \right] + \mathcal{O}_\Xi(\Psi^4).$$

The fourth term and the fifth term in (7.5) require *Step 3* only, and we can easily see that

$$\frac{2}{(\lambda v_a - \tau)^3} \mathbb{E} \left[ (z + m^{(a)} - \tau) \frac{1}{N} \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} \right] = \frac{2}{(\lambda v_a - \tau)^3} \mathbb{E} \left[ (z + m - \tau) \frac{1}{N} \sum_s G_{is} G_{si} \right] + \mathcal{O}_\Xi(\Psi^4)$$

and

$$\frac{4}{(\lambda v_a - \tau)^3} \mathbb{E} \left[ \frac{1}{N^2} \sum_{p,q,s}^{(a)} G_{ip}^{(a)} G_{pq}^{(a)} G_{qi}^{(a)} \right] = \frac{4}{(\lambda v_a - \tau)^3} \mathbb{E} \left[ \frac{1}{N^2} \sum_{p,q,s} G_{ip} G_{pq} G_{qi} \right] + \mathcal{O}_\Xi(\Psi^4).$$

Thus, we now have from (7.5) that

$$\begin{aligned} \mathbb{E}[G_{ia} G_{ai}] &= \frac{1}{(\lambda v_a - \tau)^2} \mathbb{E} \left[ \frac{1}{N} \sum_s G_{is} G_{si} \right] + \frac{2}{(\lambda v_a - \tau)^3} \mathbb{E} \left[ \frac{1}{N^2} \sum_{s,k} G_{ik} G_{ks} G_{si} \right] \\ &\quad + \frac{2}{(\lambda v_a - \tau)^3} \mathbb{E} \left[ (z + m - \tau) \frac{1}{N} \sum_s G_{is} G_{si} \right] + \mathcal{O}_\Xi(\Psi^4), \end{aligned} \quad (7.6)$$

where every non-negligible term is fully expanded. We remark that the coefficients of the last two terms in (7.6), which are of  $\mathcal{O}_\Xi(\Psi^3)$  by a naive power-counting, contain the same factor  $(\lambda v_a - \tau)^{-3}$ . This is not a mere coincidence but an intrinsic structure of the procedure.

**7.2. Proof of Equation (7.2).** From (7.6), we find that

$$\begin{aligned} &\sum_a \left( -\dot{\lambda} v_a \mathbb{E}[G_{ia} G_{ai}] + \dot{z} \mathbb{E}[G_{ia} G_{ai}] \right) \\ &= -\dot{\lambda} \left( \frac{1}{N} \sum_a \frac{v_a}{(\lambda v_a - \tau)^2} \right) \mathbb{E} \left[ \sum_s G_{is} G_{si} \right] + \dot{z} \mathbb{E} \left[ \sum_s G_{is} G_{si} \right] - \dot{\lambda} v_i \mathbb{E}[(G_{ii})^2] \\ &\quad - \dot{\lambda} \left( \frac{1}{N} \sum_a \frac{2v_a}{(\lambda v_a - \tau)^3} \right) \mathbb{E} \left[ (z + m - \tau) \sum_s G_{is} G_{si} + \frac{1}{N} \sum_{s,k} G_{ik} G_{ks} G_{si} \right] + \mathcal{O}_\Xi(N^{-\delta} \Psi), \end{aligned} \quad (7.7)$$

where we use that  $\dot{\lambda} = O(N^{-\delta})$  and that the contribution from the case  $a = i$  in the summation is negligible.

From the explicit computation of  $\dot{z}$  (with  $\gamma \equiv 1$ ) in (6.12), we see that the first two terms on the right side of (7.7) add up to zero. After taking the imaginary part of the third term, we find from the local law that

$$\text{Im } \dot{\lambda} v_i \mathbb{E}[G_{ii}^2] = 2\dot{\lambda} v_i \mathbb{E}[\text{Re } G_{ii} \cdot \text{Im } G_{ii}] = \mathcal{O}_\Xi(N^{-\delta} \Psi).$$

Thus, in order to prove (7.2), it suffices to show that

$$\mathbb{E} \left[ (z + m - \tau) \sum_s G_{is} G_{si} + \frac{1}{N} \sum_{s,k} G_{ik} G_{ks} G_{si} \right] = \mathcal{O}_\Xi(\Psi). \quad (7.8)$$

We remark that the naive size of the left side of (7.8) obtained by power counting is  $\mathcal{O}_\Xi(1)$ , hence the estimate (7.8) is non-trivial. This type of estimate will be referred to as “optical theorem” in the sequel.

To prove (7.8), we go back to (7.6). After summing over  $a$ , we have

$$\begin{aligned} \mathbb{E} \left[ \frac{1}{N} \sum_a G_{ia} G_{ai} \right] &= \left( \frac{1}{N} \sum_a \frac{1}{(\lambda v_a - \tau)^2} \right) \mathbb{E} \left[ \frac{1}{N} \sum_s G_{is} G_{si} \right] \\ &\quad + \left( \frac{2}{N} \sum_a \frac{1}{(\lambda v_a - \tau)^3} \right) \mathbb{E} \left[ (z + m - \tau) \frac{1}{N} \sum_s G_{is} G_{si} + \frac{1}{N^2} \sum_{s,k} G_{ik} G_{ks} G_{si} \right] + \mathcal{O}_\Xi(\Psi^4). \end{aligned}$$

Recall that we have set  $\gamma = 1$ , hence

$$\frac{1}{N} \sum_a \frac{1}{(\lambda v_a - \tau)^2} = 1, \quad (7.9)$$

and we obtain the following non-trivial estimate,

$$\left( \frac{2}{N} \sum_a \frac{1}{(\lambda v_a - \tau)^3} \right) \mathbb{E} \left[ (z + m - \tau) \frac{1}{N} \sum_s G_{is} G_{si} + \frac{1}{N^2} \sum_{s,k} G_{ik} G_{ks} G_{si} \right] = \mathcal{O}_\Xi(\Psi^4). \quad (7.10)$$

Since the coefficient  $N^{-1} \sum_a (\lambda v_a - \tau)^{-3}$  is bounded uniformly away from zero on  $\Xi$ , we find that the optical theorem (7.8) indeed holds. This completes the proof of Equation (7.2).

To conclude this section, we mention that the “optical theorem” is a consequence of the “sum rule” (7.9): In the expansion of  $\sum_a G_{ia} G_{ai}$ , for  $z$  close to the spectral edge, the leading terms cancel due to (7.9). In the bulk of the spectrum, the leading terms do not cancel but the expansion can be used to obtain optimal bounds on the average  $\frac{1}{N} \sum_a G_{ia} G_{ai}$  in the bulk of the spectrum. This mechanism has been studied in details for banded Wigner matrices in [16].

## 8. PROOF OF PROPOSITION 6.1

In this section, we prove Proposition 6.1 using the following result.

**Lemma 8.1.** *For  $n \in \mathbb{N}$ , let*

$$A_n := \frac{1}{N} \sum_{j=1}^N \frac{1}{(\lambda \gamma v_j - \tau)^n}, \quad A'_n := \frac{1}{N} \sum_{j=1}^N \frac{v_j}{(\lambda \gamma v_j - \tau)^n}. \quad (8.1)$$

*Let  $H(t)$  be the solution to (6.2) with initial condition  $H(0) = H_0$ ; see (6.1). Let  $F$  be a smooth function satisfying (5.7) and let  $\mathfrak{X}$  be given by (6.6). Let  $\Theta \equiv \Theta(t, H(t))$  denote the function in Proposition 6.1.*

*Then there exist random variables  $X_2 \equiv X_2(z)$  and  $X_3 \equiv X_3(z)$  with  $X_2 = \mathcal{O}_\Xi(\Psi^2)$  and  $X_3 = \mathcal{O}_\Xi(\Psi^3)$  such that*

$$\mathbb{E}[\Theta] = N \operatorname{Im} \int_{E_1-2}^{E_2-2} \mathbb{E} [C_2 N X_2 + C_3 N X_3 + C_0 F'(\mathfrak{X}) + C'_0 F'(\mathfrak{X})(z + \gamma^2 m - \tau)] dy + \mathcal{O}_\Xi(N^{1/2} \Psi^2), \quad (8.2)$$

*uniformly in  $t \geq 0$ , where we use the notation  $z \equiv z(t) = \widehat{L}_+(t) + y + i\eta$ .*

*The coefficients  $C_2, C_3, C_0, C'_0$  in (8.2) are functions of  $t$  and  $\widehat{v}$  only that are explicitly given by*

$$C_2 = -\partial_t(\lambda \gamma) \gamma^2 A'_2 + \dot{z} + 2\dot{\gamma} \gamma A_1, \quad (8.3)$$

$$C_3 = 2\gamma^2 \left( -\partial_t(\lambda \gamma) \left( A'_3 - \frac{A_3 A'_4}{A_4} \right) + \dot{\gamma} \gamma^{-1} \left( \gamma^{-2} - \frac{2A_3^2}{A_4} \right) \right), \quad (8.4)$$

$$C_0 = -\partial_t(\lambda \gamma) \left( A'_2 - \frac{A_2 A'_4}{A_4} \right) - 2\dot{\gamma} \gamma \frac{A_3}{\gamma^2 A_4}, \quad (8.5)$$

$$C'_0 = -\partial_t(\lambda \gamma) \left( A'_3 - \frac{A_3 A'_4}{A_4} \right) + \dot{\gamma} \gamma^{-1} \left( \gamma^{-2} - \frac{2A_3^2}{A_4} \right), \quad (8.6)$$

*with  $\dot{z} = \partial_t \widehat{L}_+(t)$ .*

**Remark 8.2.** The random variable  $X_2$  in Lemma 8.1 is defined by

$$X_2 := \frac{1}{N^2} \sum_{i,s} F'(\mathfrak{X}) G_{is} G_{si}.$$

The precise definition of  $X_3$  is given in (C.2) below. (See also (A.3) and (A.4).)

**Remark 8.3.** By definition, we have for  $n \geq 2$  that

$$\lambda A'_n - \tau A_n = A_{n-1}.$$

We also have

$$A_1 = \widehat{m}_{fc}(\widehat{L}_+), \quad A_2 = \gamma^{-2}, \quad A_3 = -\gamma^6. \quad (8.7)$$

See Remark 4.5 for more details.

We prove Lemma 8.1 in the Appendix C by using the ideas demonstrated in Section 7 and estimates carried out in the Appendices A and B. Applying the three step expansion procedure of Section 7 to the right side of (6.11), we obtain the leading order term  $NX_2$ , which corresponds to the term  $\sum_a G_{ia}G_{ai}$  in Section 7. By power counting we have  $X_2 = \mathcal{O}_\Xi(\Psi^2)$ . Continuing the expansion procedure, we obtain the next order term  $X_3$ , with  $X_3 = \mathcal{O}_\Xi(\Psi^3)$ . We also compute the third order term, but it can be absorbed into  $X_3$  by using a higher order “optical theorem”, i.e., an extension of (7.8). The next higher order terms are negligible. Finally, unlike as in the expansion in Section 7, the proof of Lemma 8.1 requires estimates on the diagonal terms corresponding to the terms from the case  $i = a$  in (7.7). These terms become the sub-leading, but not negligible, terms  $F'(\mathfrak{X})$  and  $F'(\mathfrak{X})(z + \gamma^2 m - \tau)$  in (8.2).

*Proof of Proposition 6.1.* Assuming the validity of (8.2), it suffices to show that  $C_2 = C_3 = C'_0 = 0$ , since  $C_0$  and  $F'(\mathfrak{X})$  are real, hence they vanish after taking the imaginary part.

In (6.12), we showed that

$$\dot{z} = \partial_t \widehat{L}_+ = -2\dot{\gamma}\gamma A_1 + \partial_t(\lambda\gamma)\gamma^2 A'_2,$$

thus we find from (8.3) that  $C_2 = 0$ .

From the definition of  $A_2$ , we have

$$A_2 = \frac{1}{N} \sum_a \frac{1}{(\lambda\gamma v_a - \tau)^2} = \gamma^{-2}.$$

Taking the partial derivative with respect to  $t$ , we obtain

$$\partial_t(\lambda\gamma)A'_3 = \dot{\tau}A_3 + \gamma^{-3}\dot{\gamma}. \quad (8.8)$$

Similarly, from the definition of  $A_3$  we obtain

$$\partial_t(\lambda\gamma)A'_4 = \dot{\tau}A_4 - \frac{\dot{A}_3}{3}. \quad (8.9)$$

Thus, combining (8.4), (8.8) and (8.9) we get

$$\begin{aligned} C_3 &= \left( -\partial_t(\lambda\gamma) \left( A'_3 - \frac{A_3 A'_4}{A_4} \right) + \dot{\gamma}\gamma^{-1} \left( \gamma^{-2} - \frac{2A_3^2}{A_4} \right) \right) \\ &= -\dot{\tau}A_3 - \dot{\gamma}\gamma^{-3} + \frac{A_3}{A_4} \left( \dot{\tau}A_4 - \frac{\dot{A}_3}{3} \right) + \dot{\gamma}\gamma^{-1} \left( \gamma^{-2} - \frac{2A_3^2}{A_4} \right) \\ &= -\frac{A_3}{3\gamma A_4} (\gamma \dot{A}_3 + 6\dot{\gamma}A_3). \end{aligned}$$

Since  $\gamma^{-6} = -A_3$ , we obtain  $C_3 = 0$ .

Similarly, one shows that  $C'_0 = 0$  as well.

Therefore, we have shown that

$$\mathbb{E}[\Theta] = \mathcal{O}_\Xi(N^{1/2}\Psi^2),$$

which completes the proof of Proposition 6.1. □

**Remark 8.4.** From the relation

$$-\partial_t(\lambda\gamma)A'_2 + \dot{\tau}A_2 = \dot{A}_1 = \partial_t \widehat{m}_{fc}(\widehat{L}_+),$$

we conclude, together with (8.5) and the identity  $\gamma^{-6} = -A_3$ , that

$$C_0 = \partial_t \widehat{m}_{fc}(\widehat{L}_+) - \dot{\tau}A_2 + \frac{A_2}{A_4} \left( \dot{\tau}A_4 - \frac{\dot{A}_3}{3} \right) - \frac{2\dot{\gamma}A_2 A_3}{\gamma A_4} = \partial_t \widehat{m}_{fc}(\widehat{L}_+),$$

as was to be expected.

The proof of Lemma 8.1 is divided into three steps. These steps are outlined in the Appendices A, B and C.



## APPENDIX A

In a first step of the proof of Lemma 8.1, we expand in this appendix the first term on the right side of (6.11). The aim of this expansion is to decouple the deterministic part  $v_a$  from the random resolvent part by deriving an approximation of the form

$$\mathbb{E}[F'(\mathfrak{X})G_{ia}(z)G_{ai}(z)] = \sum_k \sum_l f_{k,l}(v_a) \mathbb{E}[Y_{k,l,i}(z)] + \mathcal{O}_\Xi(\Psi^5), \quad (\text{A.1})$$

for a finite family of deterministic functions  $(f_{k,l})$  and a finite family of random variables  $(Y_{k,l,i} \equiv (Y_{k,l,i}(z)))$ , both indexed by natural numbers  $l, k$  and the index  $i$  such that  $Y_{k,l,i} = \mathcal{O}_\Xi(\Psi^{k+1})$  for all  $l$  and  $i$ . In (A.1), we have implicitly chosen  $z = \widehat{L}_+ + y + i\eta$ ,  $y \in [-N^{-2/3+\epsilon}, N^{-2/3+\epsilon}]$  and we will do so hereafter. Later, we will see that it suffices to consider  $k = 1, 2, 3$ .

The purpose of the approximation in (A.1) is twofold. First, after multiplying (A.1) by  $v_a$ , we obtain the expression corresponding to (7.7). This enables us to estimate the right side of (6.11). Second, we can prove from (A.1) an optical theorem, which is essential in the proof of Lemma 8.1. (The corresponding result for small  $\lambda$  was given in (7.8).)

We remark that  $Y_{1,l,i}$  and  $Y_{2,l,i}$  can be written in terms of the  $z$ -dependent random variables

$$\mathcal{X}_{22} := \frac{1}{N} \sum_p G_{ip} G_{pi}, \quad (\text{A.2})$$

$$\mathcal{X}_{32} := (z + \gamma^2 m - \tau) \frac{1}{N} \sum_p G_{ip} G_{pi}, \quad (\text{A.3})$$

$$\mathcal{X}_{33} := \frac{1}{N^2} \sum_{p,q} G_{ip} G_{pq} G_{qi}. \quad (\text{A.4})$$

To simplify the notation slightly, we drop the subscript  $\Xi$  in  $\mathcal{O}_\Xi$ , yet we always condition on  $\Xi$ .

From the local law in (6.4), it can be checked that

$$z + \gamma^2 m - \tau = \mathcal{O}(\Psi).$$

Thus, we have the a priori bounds

$$\mathcal{X}_{22} = \mathcal{O}(\Psi^2), \quad \mathcal{X}_{32} = \mathcal{O}(\Psi^3), \quad \mathcal{X}_{33} = \mathcal{O}(\Psi^3).$$

In the expansion of (6.11), we will also use the following  $z$ -dependent random variables:

$$\mathcal{X}_{42} := (z + \gamma^2 m - \tau)^2 \frac{1}{N} \sum_a G_{ia} G_{ai}, \quad (\text{A.5})$$

$$\mathcal{X}_{43} := (z + \gamma^2 m - \tau) \frac{1}{N^2} \sum_{a,b} G_{ia} G_{ab} G_{bi}, \quad (\text{A.6})$$

$$\mathcal{X}_{44} := \frac{1}{N^3} \sum_{a,b,c} G_{ia} G_{ab} G_{bc} G_{ci}, \quad (\text{A.7})$$

$$\mathcal{X}'_{44} := \frac{1}{N^3} \sum_{a,b,c} G_{ia} G_{ai} G_{bc} G_{cb}. \quad (\text{A.8})$$

We remark that

$$\mathcal{X}_{42}, \mathcal{X}_{43}, \mathcal{X}_{44}, \mathcal{X}'_{44} = \mathcal{O}(\Psi^4).$$

Let

$$\mathfrak{X}^{(a)} := N \int_{E_1}^{E_2} \text{Im } m^{(a)}(\widehat{L}_+ + x - 2 + i\eta) dx = N \int_{E_1-2}^{E_2-2} \text{Im } m^{(a)}(\widehat{L}_+ + y + i\eta) dy. \quad (\text{A.9})$$

Since

$$m(\widehat{L}_+ + x - 2 + i\eta) - m^{(a)}(\widehat{L}_+ + x - 2 + i\eta) = \mathcal{O}(\Psi^2),$$

we have

$$\mathfrak{X} - \mathfrak{X}^{(a)} = \mathcal{O}(\Psi),$$

hence by a Taylor expansion

$$F'(\mathfrak{X}) = F'(\mathfrak{X}^{(a)}) + F''(\mathfrak{X}^{(a)})(\mathfrak{X} - \mathfrak{X}^{(a)}) + \frac{F'''(\mathfrak{X}^{(a)})}{2}(\mathfrak{X} - \mathfrak{X}^{(a)})^2 + \mathcal{O}(\Psi^3). \quad (\text{A.10})$$

In particular,

$$F'(\mathfrak{X}) - F'(\mathfrak{X}^{(a)}) = \mathcal{O}(\Psi).$$

We consider first the case  $a \neq i$ . (Later, we will add the term  $F'(\mathfrak{X})G_{ii}^2$  for the case  $a = i$ .) The general idea of the expansion is the same as in Section 7, and the ultimate goal of the expansion is to decouple the index  $a$  appearing as a lower or upper index of the resolvent entries from all other indices. In a first step, using the resolvent formula (3.9), we find

$$G_{ia}G_{ai} = G_{aa}^2 \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} = \frac{1}{(\lambda\gamma v_a - z - \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa})^2} \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)}.$$

Applying the large deviation estimates of Lemma 3.3 to the term  $\sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa}$ , we find

$$z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau = \mathcal{O}(\Psi).$$

Thus, by expanding around  $(\lambda\gamma v_a - \tau)$ , we get

$$\begin{aligned} G_{ia}G_{ai} &= \frac{1}{(\lambda\gamma v_a - \tau)^2} \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \\ &\quad + \frac{2}{(\lambda\gamma v_a - \tau)^3} \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \\ &\quad + \frac{3}{(\lambda\gamma v_a - \tau)^4} \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right)^2 \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{A.11})$$

Using (A.10), we also find

$$\begin{aligned} F'(\mathfrak{X})G_{ia}G_{ai} &= \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \\ &\quad + \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} + \frac{F'''(\mathfrak{X}^{(a)})}{2(\lambda\gamma v_a - \tau)^2} (\mathfrak{X} - \mathfrak{X}^{(a)})^2 \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \\ &\quad + \frac{2F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \\ &\quad + \frac{2F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X} - \mathfrak{X}^{(a)}) \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \\ &\quad + \frac{3F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right)^2 \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{A.12})$$

We notice that, after taking the partial expectation  $\mathbb{E}_a$ , i.e., performing *Step 2* of Section 7, the right side of (A.12) does no more contain  $a$  as a lower index. In the next step, *Step 3*, we remove the upper index  $a$  in the resolvent entries. After completing one cycle of *Steps 1-3*, we find that the index  $a$  in the leading order term is decoupled, hence the expansion for the leading term is finished. We will repeat the same procedure until we obtain an expansion where all non-negligible terms are fully expanded. In the rest of this section, we expand each term in (A.12) by following the procedure in Section 7.

**A.1. Expansion of the first term in (A.12).** We begin by taking the partial expectation  $\mathbb{E}_a$  of the first term in (A.12). This corresponds to *Step 2* in Section 7. From the relation

$$G_{is}^{(a)} G_{si}^{(a)} = G_{is} G_{si} - \frac{G_{ia} G_{as}}{G_{aa}} G_{si} - G_{is}^{(a)} \frac{G_{sa} G_{ai}}{G_{aa}},$$

we have that

$$\begin{aligned}
\mathbb{E}_a \left[ \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \right] &= \frac{\gamma^2}{N} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} \\
&= \frac{\gamma^2}{N} \frac{F'(\mathfrak{X})}{(\lambda\gamma v_a - \tau)^2} \sum_s G_{is} G_{si} - \frac{\gamma^2}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_s^{(a)} G_{is} G_{si} \\
&\quad - \frac{\gamma^2}{2N} \frac{F'''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} (\mathfrak{X} - \mathfrak{X}^{(a)})^2 \sum_s^{(a)} G_{is} G_{si} \\
&\quad - \frac{\gamma^2}{N} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} \sum_s^{(a)} \frac{G_{ia} G_{as}}{G_{aa}} G_{si} - \frac{\gamma^2}{N} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} \sum_s^{(a)} G_{is}^{(a)} \frac{G_{sa} G_{ai}}{G_{aa}} + \mathcal{O}(\Psi^5). \tag{A.13}
\end{aligned}$$

By the definition of  $\mathcal{X}_{22}$  in (A.2), we have for the first term on the right side of (A.13) that

$$\mathbb{E} \left[ \frac{\gamma^2}{N} \frac{F'(\mathfrak{X})}{(\lambda\gamma v_a - \tau)^2} \sum_s G_{is} G_{si} \right] = \frac{\gamma^2}{(\lambda\gamma v_a - \tau)^2} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{22}], \tag{A.14}$$

and we stop expanding it since  $v_a$  is already decoupled from the random part. All the other terms in (A.13) need to be expanded further. For that purpose we repeat the same procedure again. We remark that we may take the partial expectation  $\mathbb{E}_a$  as many times as we want, since for any random variable  $X$ ,

$$\mathbb{E}X = \mathbb{E} \mathbb{E}_a X = \mathbb{E} \mathbb{E}_a \mathbb{E}_a X,$$

and, although not written explicitly in (A.13), the object we expand is  $\mathbb{E}[F'(\mathfrak{X}) G_{ia} G_{ai}]$ , which has the full expectation.

A.1.1. *Expansion of the second term in (A.13).* We now expand the second term in (A.13). We begin with

$$\begin{aligned}
&\frac{\gamma^2}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_s^{(a)} G_{is} G_{si} \\
&= \frac{\gamma^2}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_s^{(a)} \left( G_{is}^{(a)} G_{si}^{(a)} + \frac{G_{ia} G_{as}}{G_{aa}} G_{si} + G_{is}^{(a)} \frac{G_{sa} G_{ai}}{G_{aa}} \right) \\
&= \frac{\gamma^2}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} + \frac{\gamma^2}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_{s,p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qs}^{(a)} G_{si} \\
&\quad + \frac{\gamma^2}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_{s,p,q}^{(a)} G_{is}^{(a)} G_{sp}^{(a)} h_{pa} h_{aq} G_{qi}^{(a)} + \mathcal{O}(\Psi^5). \tag{A.15}
\end{aligned}$$

By the definition of  $\mathfrak{X}$  and  $\mathfrak{X}^{(a)}$ , we have

$$\mathfrak{X} - \mathfrak{X}^{(a)} = \text{Im} \int_{E_1-2}^{E_2-2} \sum_j \frac{G_{ja} G_{aj} (\widehat{L}_+ + \widetilde{y} + i\eta)}{G_{aa} (\widehat{L}_+ + \widetilde{y} + i\eta)} d\widetilde{y}.$$

For simplicity, abbreviate  $\widetilde{G} \equiv \widetilde{G}(\widehat{L}_+ + y + i\eta)$ , where  $\widetilde{y} \in [E_1 - 2, E_2 - 2]$ . We then have

$$\mathfrak{X} - \mathfrak{X}^{(a)} = \text{Im} \int_{E_1-2}^{E_2-2} \sum_j \frac{\widetilde{G}_{ja} \widetilde{G}_{aj}}{\widetilde{G}_{aa}} d\widetilde{y},$$

and  $\mathfrak{X} - \mathfrak{X}^{(a)} = \mathcal{O}(\Psi)$ . Following the procedure above, we consider

$$\begin{aligned} \frac{\tilde{G}_{ja}\tilde{G}_{aj}}{\tilde{G}_{aa}} &= \tilde{G}_{aa} \sum_{p,q}^{(a)} \tilde{G}_{jp}^{(a)} h_{pa} h_{aq} \tilde{G}_{qj}^{(a)} \\ &= \frac{1}{\lambda\gamma v_a - \tau} \sum_{p,q}^{(a)} \tilde{G}_{jp}^{(a)} h_{pa} h_{aq} \tilde{G}_{qj}^{(a)} \\ &\quad + \frac{1}{(\lambda\gamma v_a - \tau)^2} \left( z + \sum_{r,s}^{(a)} h_{ar} \tilde{G}_{rs}^{(a)} h_{sa} - \tau \right) \sum_{p,q}^{(a)} \tilde{G}_{jp}^{(a)} h_{pa} h_{aq} \tilde{G}_{qj}^{(a)} + \mathcal{O}(\Psi^4). \end{aligned} \quad (\text{A.16})$$

Taking the partial expectation  $\mathbb{E}_a$ , we find that

$$\begin{aligned} \mathbb{E}_a \frac{\tilde{G}_{ja}\tilde{G}_{aj}}{\tilde{G}_{aa}} &= \frac{\gamma^2}{\lambda\gamma v_a - \tau} \frac{1}{N} \sum_p^{(a)} \tilde{G}_{jp}^{(a)} \tilde{G}_{pj}^{(a)} + \frac{\gamma^2}{(\lambda\gamma v_a - \tau)^2} \left( z + \gamma^2 \tilde{m}^{(a)} - \tau \right) \frac{1}{N} \sum_p^{(a)} \tilde{G}_{jp}^{(a)} \tilde{G}_{pj}^{(a)} \\ &\quad + \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^2} \frac{1}{N^2} \sum_{p,q}^{(a)} \tilde{G}_{jp}^{(a)} \tilde{G}_{pq}^{(a)} \tilde{G}_{qj}^{(a)} + \mathcal{O}(\Psi^4), \end{aligned} \quad (\text{A.17})$$

where we let

$$\tilde{m} := \frac{1}{N} \sum_m \tilde{G}_{mm}, \quad \tilde{m}^{(a)} := \frac{1}{N} \sum_m^{(a)} \tilde{G}_{mm}^{(a)}.$$

We define

$$\mathfrak{X}_{22} := \text{Im} \int_{E_1-2}^{E_2-2} \frac{1}{N} \sum_{j,k} \tilde{G}_{jk} \tilde{G}_{kj} d\tilde{y}, \quad (\text{A.18})$$

$$\mathfrak{X}_{32} := \text{Im} \int_{E_1-2}^{E_2-2} (z + \gamma^2 \tilde{m} - \tau) \frac{1}{N} \sum_{j,k} \tilde{G}_{jk} \tilde{G}_{kj} d\tilde{y}, \quad (\text{A.19})$$

$$\mathfrak{X}_{33} := \text{Im} \int_{E_1-2}^{E_2-2} \frac{1}{N^2} \sum_{j,k,m} \tilde{G}_{jk} \tilde{G}_{km} \tilde{G}_{mj} d\tilde{y}, \quad (\text{A.20})$$

and similarly,

$$\mathfrak{X}_{22}^{(a)} := \text{Im} \int_{E_1-2}^{E_2-2} \frac{1}{N} \sum_{j,k}^{(a)} \tilde{G}_{jk}^{(a)} \tilde{G}_{kj}^{(a)} d\tilde{y}, \quad (\text{A.21})$$

$$\mathfrak{X}_{32}^{(a)} := \text{Im} \int_{E_1-2}^{E_2-2} (z + \gamma^2 \tilde{m}^{(a)} - \tau) \frac{1}{N} \sum_{j,k}^{(a)} \tilde{G}_{jk}^{(a)} \tilde{G}_{kj}^{(a)} d\tilde{y}, \quad (\text{A.22})$$

$$\mathfrak{X}_{33}^{(a)} := \text{Im} \int_{E_1-2}^{E_2-2} \frac{1}{N^2} \sum_{j,k,m}^{(a)} \tilde{G}_{jk}^{(a)} \tilde{G}_{km}^{(a)} \tilde{G}_{mj}^{(a)} d\tilde{y}. \quad (\text{A.23})$$

We notice that

$$\mathfrak{X}_{22}, \mathfrak{X}_{22}^{(a)} = \mathcal{O}(\Psi), \quad \mathfrak{X}_{32}, \mathfrak{X}_{33}, \mathfrak{X}_{32}^{(a)}, \mathfrak{X}_{33}^{(a)} = \mathcal{O}(\Psi^2).$$

Thus, we have

$$\mathbb{E}_a[\mathfrak{X} - \mathfrak{X}^{(a)}] = \frac{\gamma^2}{\lambda\gamma v_a - \tau} \mathfrak{X}_{22}^{(a)} + \frac{\gamma^2}{(\lambda\gamma v_a - \tau)^2} \mathfrak{X}_{32}^{(a)} + \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^2} \mathfrak{X}_{33}^{(a)} + \mathcal{O}(\Psi^3), \quad (\text{A.24})$$

which also yields

$$\begin{aligned}
\mathbb{E}_a \left[ \frac{\gamma^2}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} \right] \\
= \frac{\gamma^4}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \mathfrak{X}_{22}^{(a)} \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} + \frac{\gamma^4}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \mathfrak{X}_{32}^{(a)} \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} \\
+ \frac{2\gamma^6}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \mathfrak{X}_{33}^{(a)} \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} + \mathcal{O}(\Psi^5). \tag{A.25}
\end{aligned}$$

We repeat the same procedure again for the first term in (A.25). We expand

$$\begin{aligned}
\frac{\gamma^4}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \mathfrak{X}_{22}^{(a)} \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} \\
= \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^3} F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{22} - \frac{\gamma^4}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \mathfrak{X}_{22}^{(a)} \sum_s^{(a)} G_{is}^{(a)} \frac{G_{sa} G_{ai}}{G_{aa}} - \frac{\gamma^4}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \mathfrak{X}_{22}^{(a)} \sum_s^{(a)} \frac{G_{ia} G_{as}}{G_{aa}} G_{si} \\
- \frac{\gamma^4}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X}_{22} - \mathfrak{X}_{22}^{(a)}) \sum_s^{(a)} G_{is} G_{si} - \frac{\gamma^4}{N} \frac{F'''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X} - \mathfrak{X}^{(a)}) \mathfrak{X}_{22} \sum_s^{(a)} G_{is} G_{si} + \mathcal{O}(\Psi^5), \tag{A.26}
\end{aligned}$$

where we used that

$$F''(\mathfrak{X}) = F''(\mathfrak{X}^{(a)}) + F'''(\mathfrak{X}^{(a)})(\mathfrak{X} - \mathfrak{X}^{(a)}) + \mathcal{O}(\Psi^2)$$

and that

$$\mathfrak{X}_{22} - \mathfrak{X}_{22}^{(a)} = \text{Im} \int_{E_1-2}^{E_2-2} \frac{1}{N} \sum_{j,k} \left( \frac{\tilde{G}_{ja} \tilde{G}_{ak}}{\tilde{G}_{aa}} \tilde{G}_{kj} + \tilde{G}_{jk}^{(a)} \frac{\tilde{G}_{ka} \tilde{G}_{aj}}{\tilde{G}_{aa}} \right) d\tilde{y} = \mathcal{O}(\Psi^2).$$

The first term in (A.26) is fully expanded, i.e.,  $v_a$  is decoupled from the random part. All the other terms in (A.26) are of  $\mathcal{O}(\Psi^4)$ . To a  $\mathcal{O}(\Psi^4)$ -term we can freely add or remove the upper index  $(a)$  at the expense of an error term of  $\mathcal{O}(\Psi^5)$ , which is negligible in the calculation. For example, the second term in (A.26) satisfies

$$\begin{aligned}
\mathbb{E}_a \left[ \frac{\gamma^4}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \mathfrak{X}_{22}^{(a)} \sum_s^{(a)} G_{is}^{(a)} \frac{G_{sa} G_{ai}}{G_{aa}} \right] &= \mathbb{E}_a \left[ \frac{\gamma^4}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \mathfrak{X}_{22}^{(a)} G_{aa} \sum_{s,p,q}^{(a)} G_{is}^{(a)} G_{sp}^{(a)} h_{pa} h_{aq} G_{qi}^{(a)} \right] \\
&= \frac{\gamma^6}{N^2} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \mathfrak{X}_{22}^{(a)} \sum_{s,p}^{(a)} G_{is}^{(a)} G_{sp}^{(a)} G_{pi}^{(a)} + \mathcal{O}(\Psi^5) = \frac{\gamma^6}{N^2} \frac{F''(\mathfrak{X})}{(\lambda\gamma v_a - \tau)^4} \mathfrak{X}_{22} \sum_{s,p} G_{is} G_{sp} G_{pi} + \mathcal{O}(\Psi^5) \\
&= \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{33} + \mathcal{O}(\Psi^5). \tag{A.27}
\end{aligned}$$

Similarly, for the third term in (A.26) we have

$$\mathbb{E}_a \left[ \frac{\gamma^4}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \mathfrak{X}_{22}^{(a)} \sum_s^{(a)} \frac{G_{ia} G_{as}}{G_{aa}} G_{si} \right] = \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{33} + \mathcal{O}(\Psi^5). \tag{A.28}$$

In order to control the fourth term in (A.26), we first consider  $\mathbb{E}_a(\mathfrak{X}_{22} - \mathfrak{X}_{22}^{(a)})$  as in (A.24) and get

$$\mathbb{E}_a(\mathfrak{X}_{22} - \mathfrak{X}_{22}^{(a)}) = \frac{2\gamma^2}{\lambda\gamma v_a - \tau} \mathfrak{X}_{33} + \mathcal{O}(\Psi^3). \tag{A.29}$$

We then obtain that

$$\mathbb{E}_a \left[ \frac{\gamma^4}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X}_{22} - \mathfrak{X}_{22}^{(a)}) \sum_s^{(a)} G_{is} G_{si} \right] = \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} F''(\mathfrak{X}) \mathfrak{X}_{33} \mathcal{X}_{22} + \mathcal{O}(\Psi^5). \tag{A.30}$$

Finally, the last term in (A.26) becomes

$$\mathbb{E}_a \left[ \frac{\gamma^4}{N} \frac{F'''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X} - \mathfrak{X}^{(a)}) \mathfrak{X}_{22} \sum_s^{(a)} G_{is} G_{si} \right] = \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} F'''(\mathfrak{X}) (\mathfrak{X}_{22})^2 \mathcal{X}_{22} + \mathcal{O}(\Psi^5). \quad (\text{A.31})$$

We thus have shown from (A.26), (A.27), (A.28), (A.30) and (A.31) that

$$\begin{aligned} & \mathbb{E} \left[ \frac{\gamma^4}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \mathfrak{X}_{22}^{(a)} \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} \right] \\ &= \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{22}] - \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{33}] \\ & \quad - \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{33} \mathcal{X}_{22}] - \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'''(\mathfrak{X}) (\mathfrak{X}_{22})^2 \mathcal{X}_{22}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{A.32})$$

We next return to the second and the third terms in (A.25). Since these are of  $\mathcal{O}(\Psi^4)$ , we observe that

$$\mathbb{E} \left[ \frac{\gamma^4}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \mathfrak{X}_{32}^{(a)} \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} \right] = \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{32} \mathcal{X}_{22}] + \mathcal{O}(\Psi^5) \quad (\text{A.33})$$

and that

$$\mathbb{E} \left[ \frac{2\gamma^6}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \mathfrak{X}_{33}^{(a)} \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} \right] = \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{33} \mathcal{X}_{22}] + \mathcal{O}(\Psi^5). \quad (\text{A.34})$$

We then obtain from (A.25), (A.32), (A.33) and (A.34) that

$$\begin{aligned} & \mathbb{E} \left[ \frac{\gamma^2}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} \right] \\ &= \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{22}] + \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{32} \mathcal{X}_{22}] \\ & \quad - \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{33}] - \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'''(\mathfrak{X}) (\mathfrak{X}_{22})^2 \mathcal{X}_{22}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{A.35})$$

This finishes the expansion for the first term in (A.15).

We now return to the second term in (A.15). We note that

$$\begin{aligned} & \frac{\gamma^2}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_{s,p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qs}^{(a)} G_{si} \\ &= \frac{\gamma^2}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \left( \text{Im} \int_{E_1-2}^{E_2-2} \sum_j \frac{\tilde{G}_{ja} \tilde{G}_{aj}}{\tilde{G}_{aa}} d\tilde{y} \right) \sum_{s,p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qs}^{(a)} G_{si} \\ &= \frac{\gamma^2}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \left( \text{Im} \int_{E_1-2}^{E_2-2} \sum_j \sum_{r,s}^{(a)} \tilde{G}_{jr}^{(a)} h_{ra} h_{as} \tilde{G}_{sj}^{(a)} d\tilde{y} \right) \sum_{s,p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qs}^{(a)} G_{si}^{(a)} + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{A.36})$$

Taking the partial expectation  $\mathbb{E}_a$ , we find

$$\begin{aligned} & \mathbb{E}_a \left[ \frac{\gamma^2}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_{s,p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qs}^{(a)} G_{si} \right] \\ &= \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} F''(\mathfrak{X}^{(a)}) \mathfrak{X}_{22} \mathcal{X}_{33} + \frac{2\gamma^6}{N^3} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \left( \text{Im} \int_{E_1-2}^{E_2-2} \sum_j \sum_{r,t}^{(a)} \tilde{G}_{jr}^{(a)} \tilde{G}_{tj}^{(a)} d\tilde{y} \right) \sum_s^{(a)} G_{ir}^{(a)} G_{ts}^{(a)} G_{si}^{(a)} + \mathcal{O}(\Psi^5). \end{aligned}$$



Thus, we obtain

$$\begin{aligned}
& \mathbb{E} \left[ \frac{\gamma^2}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_{s,p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qs}^{(a)} G_{si} \right] \\
&= \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{33}] \\
&+ \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,r,t,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jr} \tilde{G}_{tj} d\tilde{y} \right) G_{ir} G_{ts} G_{si} \right] + \mathcal{O}(\Psi^5). \tag{A.37}
\end{aligned}$$

The third term in (A.15) can be dealt with in a similar manner, and we get

$$\begin{aligned}
& \mathbb{E} \left[ \frac{\gamma^2}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_{s,p,q}^{(a)} G_{is}^{(a)} G_{sp}^{(a)} h_{pa} h_{aq} G_{qi}^{(a)} \right] \\
&= \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{33}] + \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,r,t,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jr} \tilde{G}_{tj} d\tilde{y} \right) G_{ir} G_{ts} G_{si} \right] + \mathcal{O}(\Psi^5). \tag{A.38}
\end{aligned}$$

So far, we have shown from (A.15), (A.35), (A.37) and (A.38) that

$$\begin{aligned}
& \mathbb{E} \left[ \frac{\gamma^2}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_s^{(a)} G_{is} G_{si} \right] \\
&= \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{22}] + \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{32} \mathcal{X}_{22}] - \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'''(\mathfrak{X}) (\mathfrak{X}_{22})^2 \mathcal{X}_{22}] \\
&+ \frac{4\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,r,t,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jr} \tilde{G}_{tj} d\tilde{y} \right) G_{ir} G_{ts} G_{si} \right] + \mathcal{O}(\Psi^5). \tag{A.39}
\end{aligned}$$

This completes the expansion of the second term in (A.13).

A.1.2. *Expansion of the third term in (A.13).* Recall that

$$\mathfrak{X} - \mathfrak{X}^{(a)} = \text{Im} \int_{E_1-2}^{E_2-2} \sum_j \frac{\tilde{G}_{ja} \tilde{G}_{aj}}{\tilde{G}_{aa}} d\tilde{y} \tag{A.40}$$

and that  $\mathfrak{X} - \mathfrak{X}^{(a)} = \mathcal{O}(\Psi)$ . Let  $\hat{G} \equiv G(\hat{L}_+ + \hat{y} + i\eta)$ , for  $\hat{y} \in [E_1 - 2, E_2 - 2]$ . Then, we may write

$$(\mathfrak{X} - \mathfrak{X}^{(a)})^2 = \left( \text{Im} \int_{E_1-2}^{E_2-2} \sum_j \frac{\tilde{G}_{ja} \tilde{G}_{aj}}{\tilde{G}_{aa}} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \sum_k \frac{\hat{G}_{ka} \hat{G}_{ak}}{\hat{G}_{aa}} d\hat{y} \right). \tag{A.41}$$

Applying the resolvent formula (3.9) to both of the integrands, we find

$$\begin{aligned}
& (\mathfrak{X} - \mathfrak{X}^{(a)})^2 \\
&= \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{aa} \sum_j \sum_{p,q}^{(a)} \tilde{G}_{jp}^{(a)} h_{pa} h_{aq} \tilde{G}_{qj}^{(a)} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \hat{G}_{aa} \sum_k \sum_{r,t} \hat{G}_{kr}^{(a)} h_{ra} h_{at} \hat{G}_{tk}^{(a)} d\hat{y} \right) \\
&= \frac{1}{(\lambda\gamma v_a - \tau)^2} \sum_{j,k} \sum_{p,q,r,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp}^{(a)} h_{pa} h_{aq} \tilde{G}_{qj}^{(a)} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \hat{G}_{kr}^{(a)} h_{ra} h_{at} \hat{G}_{tk}^{(a)} d\hat{y} \right).
\end{aligned}$$

Thus, by taking the partial expectation  $\mathbb{E}_a$ , we get

$$\begin{aligned} \mathbb{E}_a \left( \mathfrak{X} - \mathfrak{X}^{(a)} \right)^2 &= \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^2} \frac{1}{N^2} \sum_{j,k} \sum_{p,r}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp}^{(a)} \tilde{G}_{pj}^{(a)} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \hat{G}_{kr}^{(a)} \hat{G}_{rk}^{(a)} d\hat{y} \right) \\ &\quad + \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^2} \frac{1}{N^2} \sum_{j,k} \sum_{r,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jr}^{(a)} \tilde{G}_{tj}^{(a)} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \hat{G}_{kr}^{(a)} \hat{G}_{tk}^{(a)} d\hat{y} \right) + \mathcal{O}(\Psi^3), \end{aligned}$$

which implies

$$\begin{aligned} \mathbb{E}_a \left( \mathfrak{X} - \mathfrak{X}^{(a)} \right)^2 &= \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^2} (\mathfrak{X}_{22})^2 \\ &\quad + \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^2} \frac{1}{N^2} \sum_{j,k,r,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jr} \tilde{G}_{tj} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \hat{G}_{kr} \hat{G}_{tk} d\hat{y} \right) + \mathcal{O}(\Psi^3). \end{aligned}$$

We remark that the terms with the fourth or higher moments of the entries of  $H(t)$  have negligible contributions, hence we omit the details on those terms in the calculation. Thus returning to the third term in (A.13), we obtain

$$\begin{aligned} \mathbb{E} \left[ \frac{\gamma^2}{2N} \frac{F'''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} (\mathfrak{X} - \mathfrak{X}^{(a)})^2 \sum_s^{(a)} G_{is} G_{si} \right] \\ = \frac{\gamma^6}{2(\lambda\gamma v_a - \tau)^4} \mathbb{E} [F'''(\mathfrak{X})(\mathfrak{X}_{22})^2 \mathcal{X}_{22}] \\ + \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F'''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,k,r,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jr} \tilde{G}_{tj} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \hat{G}_{kr} \hat{G}_{tk} d\hat{y} \right) \mathcal{X}_{22} \right] + \mathcal{O}(\Psi^5). \quad (\text{A.42}) \end{aligned}$$

**A.1.3. Expansion of the fourth term and the fifth term in (A.13).** We first begin with the expansion

$$\frac{\gamma^2}{N} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} \sum_s^{(a)} \frac{G_{ia} G_{as}}{G_{aa}} G_{si} = \frac{\gamma^2}{N} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} \sum_s^{(a)} \frac{G_{ia} G_{as}}{G_{aa}} G_{si}^{(a)} + \frac{\gamma^2}{N} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} \sum_s^{(a)} \frac{G_{ia} G_{as}}{G_{aa}} \frac{G_{sa} G_{ai}}{G_{aa}}. \quad (\text{A.43})$$

Expanding the first term in (A.43), we get

$$\begin{aligned} \frac{\gamma^2}{N} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} \sum_s^{(a)} \frac{G_{ia} G_{as}}{G_{aa}} G_{si}^{(a)} &= \frac{\gamma^2}{N} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \sum_{s,r,t}^{(a)} G_{ir}^{(a)} h_{ra} h_{at} G_{ts}^{(a)} G_{si}^{(a)} \\ &\quad + \frac{\gamma^2}{N} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) \sum_{s,r,t}^{(a)} G_{ir}^{(a)} h_{ra} h_{at} G_{ts}^{(a)} G_{si}^{(a)} + \mathcal{O}(\Psi^5). \quad (\text{A.44}) \end{aligned}$$

Taking the partial expectation  $\mathbb{E}_a$ , we obtain

$$\begin{aligned} \mathbb{E}_a \left[ \frac{\gamma^2}{N} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \sum_{s,r,t}^{(a)} G_{ir}^{(a)} h_{ra} h_{at} G_{ts}^{(a)} G_{si}^{(a)} \right] &= \frac{\gamma^4}{N^2} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \sum_{s,r}^{(a)} G_{ir}^{(a)} G_{rs}^{(a)} G_{si}^{(a)} \\ &= \frac{\gamma^4}{N^2} \frac{F'(\mathfrak{X})}{(\lambda\gamma v_a - \tau)^3} \sum_{s,r}^{(a)} G_{ir} G_{rs} G_{si} - \frac{\gamma^4}{N^2} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \sum_{s,r}^{(a)} G_{ir}^{(a)} G_{rs}^{(a)} \frac{G_{sa} G_{ai}}{G_{aa}} - \frac{\gamma^4}{N^2} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \sum_{s,r}^{(a)} G_{ir}^{(a)} \frac{G_{ra} G_{as}}{G_{aa}} G_{si} \\ &\quad - \frac{\gamma^4}{N^2} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \sum_{s,r}^{(a)} \frac{G_{ia} G_{ar}}{G_{aa}} G_{rs} G_{si} - \frac{\gamma^4}{N^2} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_{s,r}^{(a)} G_{ir} G_{rs} G_{si} + \mathcal{O}(\Psi^5). \quad (\text{A.45}) \end{aligned}$$

Taking the full expectation, the first term in (A.45) becomes

$$\mathbb{E} \left[ \frac{\gamma^4}{N^2} \frac{F'(\mathfrak{X})}{(\lambda\gamma v_a - \tau)^3} \sum_{s,r}^{(a)} G_{ir} G_{rs} G_{si} \right] = \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{33}] + \mathcal{O}(\Psi^5). \quad (\text{A.46})$$

For the second term in (A.45), we have that

$$\begin{aligned} \mathbb{E} \left[ \frac{\gamma^4}{N^2} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \sum_{s,r}^{(a)} G_{ir}^{(a)} G_{rs}^{(a)} \frac{G_{sa} G_{ai}}{G_{aa}} \right] &= \mathbb{E} \left[ \frac{\gamma^4}{N^2} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \sum_{s,r}^{(a)} G_{ir}^{(a)} G_{rs}^{(a)} \mathbb{E}_a \sum_{p,q}^{(a)} G_{sp}^{(a)} h_{pa} h_{aq} G_{qi}^{(a)} \right] + \mathcal{O}(\Psi^5) \\ &= \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{44}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{A.47})$$

Similarly, we can also obtain

$$\mathbb{E} \left[ \frac{\gamma^4}{N^2} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \sum_{s,r}^{(a)} G_{ir}^{(a)} \frac{G_{ra} G_{as}}{G_{aa}} G_{si} \right] = \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{44}] + \mathcal{O}(\Psi^5) \quad (\text{A.48})$$

and

$$\mathbb{E} \left[ \frac{\gamma^4}{N^2} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \sum_{s,r}^{(a)} \frac{G_{ia} G_{ar}}{G_{aa}} G_{rs} G_{si} \right] = \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{44}] + \mathcal{O}(\Psi^5). \quad (\text{A.49})$$

For the last term in (A.45), we use (A.24) to obtain that

$$\mathbb{E} \left[ \frac{\gamma^4}{N^2} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_{s,r}^{(a)} G_{ir} G_{rs} G_{si} \right] = \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X})\mathfrak{X}_{22}\mathcal{X}_{33}] + \mathcal{O}(\Psi^5). \quad (\text{A.50})$$

Thus, from (A.45), (A.46), (A.47), (A.48), (A.49) and (A.50), we find that

$$\begin{aligned} \mathbb{E} \left[ \frac{\gamma^2}{N} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \sum_{s,r,t}^{(a)} G_{ir}^{(a)} h_{ra} h_{at} G_{ts}^{(a)} G_{si}^{(a)} \right] \\ = \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{33}] - \frac{3\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{44}] - \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X})\mathfrak{X}_{22}\mathcal{X}_{33}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{A.51})$$

The second term in (A.44) satisfies

$$\begin{aligned} \mathbb{E}_a \left[ \frac{\gamma^2}{N} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) \sum_{s,r,t}^{(a)} G_{ir}^{(a)} h_{ra} h_{at} G_{ts}^{(a)} G_{si}^{(a)} \right] \\ = \frac{\gamma^4}{N^2} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \left( z + \gamma^2 m^{(a)} - \tau \right) \sum_{s,t}^{(a)} G_{it}^{(a)} G_{ts}^{(a)} G_{si}^{(a)} + \frac{2\gamma^6}{N^3} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \sum_{s,r,t}^{(a)} G_{ir}^{(a)} G_{rt}^{(a)} G_{ts}^{(a)} G_{si}^{(a)} + \mathcal{O}(\Psi^5). \end{aligned}$$

Thus, we get

$$\begin{aligned} \mathbb{E} \left[ \frac{\gamma^2}{N} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) \sum_{s,r,t}^{(a)} G_{ir}^{(a)} h_{ra} h_{at} G_{ts}^{(a)} G_{si}^{(a)} \right] \\ = \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{43}] + \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{44}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{A.52})$$

We then have from (A.44), (A.51) and (A.52) that

$$\begin{aligned} \mathbb{E} \left[ \frac{\gamma^2}{N} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} \sum_s^{(a)} \frac{G_{ia} G_{as}}{G_{aa}} G_{si}^{(a)} \right] \\ = \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{33}] + \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{43}] \\ - \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{44}] - \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X})\mathfrak{X}_{22}\mathcal{X}_{33}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{A.53})$$

Expanding the second term in (A.43), we have

$$\begin{aligned}
& \mathbb{E} \left[ \frac{\gamma^2}{N} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} \sum_s^{(a)} \frac{G_{ia} G_{as}}{G_{aa}} \frac{G_{sa} G_{ai}}{G_{aa}} \right] \\
&= \mathbb{E} \left[ \frac{\gamma^2}{N} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \sum_s^{(a)} \sum_{p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qs}^{(a)} \sum_{r,t}^{(a)} G_{ir}^{(a)} h_{ra} h_{at} G_{ts}^{(a)} \right] + \mathcal{O}(\Psi^5) \\
&= \mathbb{E} \left[ \frac{2\gamma^6}{N^3} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \sum_{s,p,r}^{(a)} G_{ip}^{(a)} G_{ps}^{(a)} G_{ir}^{(a)} G_{rs}^{(a)} \right] + \mathbb{E} \left[ \frac{\gamma^6}{N^3} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \sum_{s,r,t}^{(a)} G_{ir}^{(a)} G_{ts}^{(a)} G_{ir}^{(a)} G_{ts}^{(a)} \right] + \mathcal{O}(\Psi^5) \\
&= \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{44}] + \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}'_{44}] + \mathcal{O}(\Psi^5). \tag{A.54}
\end{aligned}$$

Thus, from (A.43), (A.53) and (A.54), we get

$$\begin{aligned}
& \mathbb{E} \left[ \frac{\gamma^2}{N} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} \sum_s^{(a)} \frac{G_{ia} G_{as}}{G_{aa}} G_{si} \right] \\
&= \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{33}] + \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{43}] + \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{44}] \\
&\quad + \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}'_{44}] - \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{33}] + \mathcal{O}(\Psi^5). \tag{A.55}
\end{aligned}$$

The fifth term in (A.13) can be expanded analogously to (A.53), and we find that

$$\begin{aligned}
& \mathbb{E} \left[ \frac{\gamma^2}{N} \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} \sum_s^{(a)} G_{is}^{(a)} \frac{G_{sa} G_{ai}}{G_{aa}} \right] \\
&= \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{33}] + \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{43}] \\
&\quad - \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{44}] - \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{33}] + \mathcal{O}(\Psi^5). \tag{A.56}
\end{aligned}$$

This finishes the expansion of the fourth term and the fifth term in (A.13).

A.1.4. *Expansion of the first term in (A.12).* Combining (A.13), (A.14), (A.39), (A.42), (A.55) and (A.56), we conclude that

$$\begin{aligned}
& \mathbb{E} \left[ \frac{F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \right] \\
&= \frac{\gamma^2}{(\lambda\gamma v_a - \tau)^2} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{22}] - \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{33}] - \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{22}] \\
&\quad - \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{32} \mathcal{X}_{22}] + \frac{\gamma^6}{2(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'''(\mathfrak{X}) (\mathfrak{X}_{22})^2 \mathcal{X}_{22}] \\
&\quad - \frac{4\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,r,t,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jr} \tilde{G}_{tj} d\tilde{y} \right) G_{ir} G_{ts} G_{si} \right] \\
&\quad - \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F'''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,k,r,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jr} \tilde{G}_{tj} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \hat{G}_{kr} \hat{G}_{tk} d\hat{y} \right) \mathcal{X}_{22} \right] \\
&\quad - \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{43}] - \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}'_{44}] + \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{33}] + \mathcal{O}(\Psi^5). \tag{A.57}
\end{aligned}$$

This finishes the expansion of the first term in (A.12).

**A.2. Expansion of the second term in (A.12).** Following the expansion procedure for the first term in (A.12), we now expand the second term. By the definitions of  $\mathfrak{X}$  and  $\mathfrak{X}^{(a)}$ , we have

$$\begin{aligned}
& \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \\
&= \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} \left( \text{Im} \int_{E_1-2}^{E_2-2} \sum_j \frac{\tilde{G}_{ja} \tilde{G}_{aj}}{\tilde{G}_{aa}} d\tilde{y} \right) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \\
&= \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \left( \text{Im} \int_{E_1-2}^{E_2-2} \sum_j \sum_{p,q}^{(a)} \tilde{G}_{jp}^{(a)} h_{pa} h_{aq} \tilde{G}_{qj}^{(a)} d\tilde{y} \right) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \\
&\quad + \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \left( \text{Im} \int_{E_1-2}^{E_2-2} \left( z + \sum_{k,m}^{(a)} h_{ak} \tilde{G}_{km}^{(a)} h_{ma} - \tau \right) \sum_j \sum_{p,q}^{(a)} \tilde{G}_{jp}^{(a)} h_{pa} h_{aq} \tilde{G}_{qj}^{(a)} d\tilde{y} \right) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \\
&\quad + \mathcal{O}(\Psi^5). \tag{A.58}
\end{aligned}$$

**A.2.1. Expansion of the first term in (A.58).** After taking the partial expectation  $\mathbb{E}_a$ , the first term on the right side of (A.58) becomes

$$\begin{aligned}
& \mathbb{E}_a \left[ \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \left( \text{Im} \int_{E_1-2}^{E_2-2} \sum_j \sum_{p,q}^{(a)} \tilde{G}_{jp}^{(a)} h_{pa} h_{aq} \tilde{G}_{qj}^{(a)} d\tilde{y} \right) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \right] \\
&= \frac{\gamma^4}{N^2} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \mathfrak{X}_{22}^{(a)} \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} + \frac{2\gamma^4}{N^2} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \sum_{j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} \tilde{G}_{tj}^{(a)} d\tilde{y} \right) G_{is}^{(a)} G_{ti}^{(a)}. \tag{A.59}
\end{aligned}$$

The expectation of the first term in (A.59) is calculated in (A.32), which is

$$\begin{aligned}
& \mathbb{E} \left[ \frac{\gamma^4}{N} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \mathfrak{X}_{22}^{(a)} \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} \right] \\
&= \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{22}] - \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{33}] \\
&\quad - \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{33} \mathcal{X}_{22}] - \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'''(\mathfrak{X}) (\mathfrak{X}_{22})^2 \mathcal{X}_{22}] + \mathcal{O}(\Psi^5). \tag{A.60}
\end{aligned}$$

We now consider the second term in (A.59). We notice that

$$\begin{aligned}
& \frac{2\gamma^4}{N^2} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \sum_{j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} \tilde{G}_{tj}^{(a)} d\tilde{y} \right) G_{is}^{(a)} G_{ti}^{(a)} \\
&= \frac{2\gamma^4}{N^2} \frac{F''(\mathfrak{X})}{(\lambda\gamma v_a - \tau)^3} \sum_{j,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{tj} d\tilde{y} \right) G_{is} G_{ti} \\
&\quad - \frac{2\gamma^4}{N^2} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \sum_{j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} \tilde{G}_{tj}^{(a)} d\tilde{y} \right) \left( G_{is}^{(a)} \frac{G_{ta} G_{ai}}{G_{aa}} + \frac{G_{ia} G_{si}}{G_{aa}} G_{ti} \right) \\
&\quad - \frac{2\gamma^4}{N^2} \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \sum_{j,s,t}^{(a)} \left[ \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} \frac{\tilde{G}_{ta} \tilde{G}_{aj}}{\tilde{G}_{aa}} d\tilde{y} \right) + \left( \text{Im} \int_{E_1-2}^{E_2-2} \frac{\tilde{G}_{ja} \tilde{G}_{as}}{\tilde{G}_{aa}} \tilde{G}_{tj} d\tilde{y} \right) \right] G_{is} G_{ti} \\
&\quad - \frac{2\gamma^4}{N^2} \frac{F'''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_{j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} \tilde{G}_{tj}^{(a)} d\tilde{y} \right) G_{is} G_{ti} + \mathcal{O}(\Psi^5). \tag{A.61}
\end{aligned}$$

We proceed as in the expansion of (A.60) to obtain

$$\begin{aligned}
& \mathbb{E} \left[ \frac{2\gamma^4}{N^2 (\lambda\gamma v_a - \tau)^3} \sum_{j,s,t}^{(a)} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} \tilde{G}_{tj}^{(a)} d\tilde{y} \right) G_{is}^{(a)} G_{ti}^{(a)} \right] \\
&= \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,s,t} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{tj} d\tilde{y} \right) G_{is} G_{ti} \right] \\
&\quad - \frac{4\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,s,t,r} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{tj} d\tilde{y} \right) G_{is} G_{tr} G_{ri} \right] \\
&\quad - \frac{4\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,s,t,r} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jr} \tilde{G}_{rs} \tilde{G}_{tj} d\tilde{y} \right) G_{is} G_{ti} \right] \\
&\quad - \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F'''(\mathfrak{X}) \mathfrak{X}_{22} \frac{1}{N^2} \sum_{j,s,t} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{tj} d\tilde{y} \right) G_{is} G_{ti} \right] + \mathcal{O}(\Psi^5). \tag{A.62}
\end{aligned}$$

A.2.2. *Expansion of the second term in (A.58).* We first notice that the second term in (A.58) is of  $\mathcal{O}(\Psi^4)$ . Taking the partial expectation  $\mathbb{E}_a$ , we get

$$\begin{aligned}
& \mathbb{E}_a \left[ \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \left( z + \sum_{k,m}^{(a)} h_{ak} \tilde{G}_{km}^{(a)} h_{ma} - \tau \right) \sum_j^{(a)} \sum_{p,q}^{(a)} \tilde{G}_{jp}^{(a)} h_{pa} h_{aq} \tilde{G}_{qj}^{(a)} d\tilde{y} \right) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \right] \\
&= \frac{\gamma^4}{N^2} \sum_{j,p,s}^{(a)} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} (z + \gamma^2 \tilde{m}^{(a)} - \tau) \tilde{G}_{jp}^{(a)} \tilde{G}_{pj}^{(a)} d\tilde{y} \right) G_{is}^{(a)} G_{si}^{(a)} \\
&\quad + \frac{2\gamma^6}{N^3} \sum_{j,p,q,s}^{(a)} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp}^{(a)} \tilde{G}_{pq}^{(a)} \tilde{G}_{qj}^{(a)} d\tilde{y} \right) G_{is}^{(a)} G_{si}^{(a)} + \frac{2\gamma^6}{N^3} \sum_{j,p,s,t}^{(a)} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{st}^{(a)} \tilde{G}_{jp}^{(a)} \tilde{G}_{pj}^{(a)} d\tilde{y} \right) G_{is}^{(a)} G_{ti}^{(a)} \\
&\quad + \frac{2\gamma^4}{N^2} \sum_{j,s,t}^{(a)} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} (z + \gamma^2 \tilde{m}^{(a)} - \tau) \tilde{G}_{js}^{(a)} \tilde{G}_{tj}^{(a)} d\tilde{y} \right) G_{is}^{(a)} G_{ti}^{(a)} \\
&\quad + \frac{8\gamma^6}{N^3} \sum_{j,p,s,t}^{(a)} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{ps}^{(a)} \tilde{G}_{jp}^{(a)} \tilde{G}_{tj}^{(a)} d\tilde{y} \right) G_{is}^{(a)} G_{ti}^{(a)} + \mathcal{O}(\Psi^5). \tag{A.63}
\end{aligned}$$

Thus, we have

$$\begin{aligned}
& \mathbb{E} \left[ \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \left( z + \sum_{k,m}^{(a)} h_{ak} \tilde{G}_{km}^{(a)} h_{ma} - \tau \right) \sum_j^{(a)} \sum_{p,q}^{(a)} \tilde{G}_{jp}^{(a)} h_{pa} h_{aq} \tilde{G}_{qj}^{(a)} d\tilde{y} \right) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \right] \\
&= \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{32} \mathcal{X}_{22}] + \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{33} \mathcal{X}_{22}] \\
&\quad + \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,s,t} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} (z + \gamma^2 \tilde{m} - \tau) \tilde{G}_{js} \tilde{G}_{tj} d\tilde{y} \right) G_{is} G_{ti} \right] \\
&\quad + \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,p,s,t} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{st} \tilde{G}_{jp} \tilde{G}_{pj} d\tilde{y} \right) G_{is} G_{ti} \right] \\
&\quad + \frac{8\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,p,s,t} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{ps} \tilde{G}_{jp} \tilde{G}_{tj} d\tilde{y} \right) G_{is} G_{ti} \right] + \mathcal{O}(\Psi^5). \tag{A.64}
\end{aligned}$$

A.2.3. *Expansion of the second term in (A.12).* We have from (A.58), (A.59), (A.60), (A.62) and (A.64) that

$$\begin{aligned}
& \mathbb{E} \left[ \frac{F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^2} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \right] \\
&= \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{22}] - \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{33}] \\
&\quad + \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'''(\mathfrak{X}) \mathfrak{X}_{32} \mathcal{X}_{22}] - \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'''(\mathfrak{X}) (\mathfrak{X}_{22})^2 \mathcal{X}_{22}] \\
&\quad + \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{tj} d\tilde{y} \right) G_{is} G_{ti} \right] \\
&\quad + \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} (z + \gamma^2 \tilde{m} - \tau) \tilde{G}_{js} \tilde{G}_{tj} d\tilde{y} \right) G_{is} G_{ti} \right] \\
&\quad - \frac{4\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,s,t,r} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{tj} d\tilde{y} \right) G_{is} G_{tr} G_{ri} \right] \\
&\quad + \frac{4\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,s,t,r} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jr} \tilde{G}_{rs} \tilde{G}_{tj} d\tilde{y} \right) G_{is} G_{ti} \right] \\
&\quad - \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F'''(\mathfrak{X}) \mathfrak{X}_{22} \frac{1}{N^2} \sum_{j,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{tj} d\tilde{y} \right) G_{is} G_{ti} \right] \\
&\quad + \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,p,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{st} \tilde{G}_{jp} \tilde{G}_{pj} d\tilde{y} \right) G_{is} G_{ti} \right] + \mathcal{O}(\Psi^5). \tag{A.65}
\end{aligned}$$

A.3. **Expansion of the third term in (A.12).** We first notice that the third term in (A.12) is of  $\mathcal{O}(\Psi^4)$ . Using (A.41) we get

$$\begin{aligned}
& \frac{F'''(\mathfrak{X}^{(a)})}{2(\lambda\gamma v_a - \tau)^2} (\mathfrak{X} - \mathfrak{X}^{(a)})^2 \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \\
&= \frac{F'''(\mathfrak{X}^{(a)})}{2(\lambda\gamma v_a - \tau)^2} \left( \text{Im} \int_{E_1-2}^{E_2-2} \sum_j \frac{\tilde{G}_{ja} \tilde{G}_{aj}}{\tilde{G}_{aa}} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \sum_k \frac{\hat{G}_{ka} \hat{G}_{ak}}{\hat{G}_{aa}} d\hat{y} \right) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \tag{A.66}
\end{aligned}$$

Taking the partial expectation  $\mathbb{E}_a$ , we find

$$\begin{aligned}
& \mathbb{E}_a \left[ \frac{F'''(\mathfrak{X}^{(a)})}{2(\lambda\gamma v_a - \tau)^2} (\mathfrak{X} - \mathfrak{X}^{(a)})^2 \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \right] \\
&= \frac{F'''(\mathfrak{X}^{(a)})}{2(\lambda\gamma v_a - \tau)^4} \frac{\gamma^6}{N^3} \sum_{j,k,p,q}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp}^{(a)} \tilde{G}_{pq}^{(a)} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \hat{G}_{kq}^{(a)} \hat{G}_{qk}^{(a)} d\hat{y} \right) \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} \\
&\quad + \frac{F'''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \frac{\gamma^6}{N^3} \sum_{j,k,p,q}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp}^{(a)} \tilde{G}_{qj}^{(a)} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \hat{G}_{kp}^{(a)} \hat{G}_{qk}^{(a)} d\hat{y} \right) \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} \\
&\quad + \frac{2F'''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \frac{\gamma^6}{N^3} \sum_{j,k,p,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp}^{(a)} \tilde{G}_{pj}^{(a)} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \hat{G}_{ks}^{(a)} \hat{G}_{tk}^{(a)} d\hat{y} \right) G_{is}^{(a)} G_{ti}^{(a)} \\
&\quad + \frac{4F'''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \frac{\gamma^6}{N^3} \sum_{j,k,p,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp}^{(a)} \tilde{G}_{sj}^{(a)} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \hat{G}_{kp}^{(a)} \hat{G}_{tk}^{(a)} d\hat{y} \right) G_{is}^{(a)} G_{ti}^{(a)} + \mathcal{O}(\Psi^5). \tag{A.67}
\end{aligned}$$



Thus, after taking the full expectation, we obtain

$$\begin{aligned}
& \mathbb{E} \left[ \frac{F'''(\mathfrak{X}^{(a)})}{2(\lambda\gamma v_a - \tau)^2} (\mathfrak{X} - \mathfrak{X}^{(a)})^2 \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \right] \\
&= \frac{\gamma^6}{2(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'''(\mathfrak{X})(\mathfrak{X}_{22})^2 \mathcal{X}_{22}] \\
&+ \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F'''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,k,p,q} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp} \tilde{G}_{qj} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \hat{G}_{kp} \hat{G}_{qk} d\hat{y} \right) \mathcal{X}_{22} \right] \\
&+ \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F'''(\mathfrak{X}) \mathfrak{X}_{22} \frac{1}{N^2} \sum_{k,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{kp} \tilde{G}_{qk} d\tilde{y} \right) G_{is} G_{ti} \right] \\
&+ \frac{4\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F'''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,k,p,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp} \tilde{G}_{sj} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \hat{G}_{kp} \hat{G}_{tk} d\hat{y} \right) G_{is} G_{ti} \right] \\
&+ \mathcal{O}(\Psi^5). \tag{A.68}
\end{aligned}$$

**A.4. Expansion of the fourth term in (A.12).** We again begin by taking the partial expectation  $\mathbb{E}_a$ . We have

$$\begin{aligned}
& \mathbb{E}_a \left[ \frac{2F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \right] \\
&= \frac{\gamma^2}{N} \frac{2F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (z + \gamma^2 m^{(a)} - \tau) \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} + \frac{\gamma^4}{N^2} \frac{4F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \sum_{s,t}^{(a)} G_{is}^{(a)} G_{st}^{(a)} G_{ti}^{(a)} + \mathcal{O}(\Psi^5). \tag{A.69}
\end{aligned}$$

Following the procedure in (A.45) we expand (A.69). We first consider

$$\begin{aligned}
& \frac{\gamma^2}{N} \frac{2F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (z + \gamma^2 m^{(a)} - \tau) \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} \\
&= \frac{\gamma^2}{N} \frac{2F'(\mathfrak{X})}{(\lambda\gamma v_a - \tau)^3} (z + \gamma^2 m - \tau) \sum_s^{(a)} G_{is} G_{si} - \frac{\gamma^2}{N} \frac{2F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (z + \gamma^2 m^{(a)} - \tau) \sum_s^{(a)} G_{is}^{(a)} \frac{G_{sa} G_{ai}}{G_{aa}} \\
&- \frac{\gamma^2}{N} \frac{2F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (z + \gamma^2 m^{(a)} - \tau) \sum_s^{(a)} \frac{G_{ia} G_{as}}{G_{aa}} G_{si} - \frac{\gamma^4}{N} \frac{2F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (m - m^{(a)}) \sum_s^{(a)} G_{is} G_{si} \\
&- \frac{\gamma^2}{N} \frac{2F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X} - \mathfrak{X}^{(a)}) (z + \gamma^2 m - \tau) \sum_s^{(a)} G_{is} G_{si} + \mathcal{O}(\Psi^5). \tag{A.70}
\end{aligned}$$

We stop expanding the first term and observe that

$$\mathbb{E} \left[ \frac{\gamma^2}{N} \frac{2F'(\mathfrak{X})}{(\lambda\gamma v_a - \tau)^3} (z + \gamma^2 m - \tau) \sum_s^{(a)} G_{is} G_{si} \right] = \frac{2\gamma^2}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{32}] + \mathcal{O}(\Psi^5). \tag{A.71}$$

We notice that all other terms in (A.70) are of  $\mathcal{O}(\Psi^4)$ . As in the estimates (A.47) and (A.48), we find for the second and the third term of (A.70) that

$$\mathbb{E} \left[ \frac{\gamma^2}{N} \frac{2F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (z + \gamma^2 m^{(a)} - \tau) \sum_s^{(a)} G_{is}^{(a)} \frac{G_{sa} G_{ai}}{G_{aa}} \right] = \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{43}] + \mathcal{O}(\Psi^5), \tag{A.72}$$

respectively,

$$\mathbb{E} \left[ \frac{\gamma^2}{N} \frac{2F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (z + \gamma^2 m^{(a)} - \tau) \sum_s^{(a)} \frac{G_{ia} G_{as}}{G_{aa}} G_{si} \right] = \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{43}] + \mathcal{O}(\Psi^5). \tag{A.73}$$

Since

$$m - m^{(a)} = \frac{1}{N} \sum_r \frac{G_{ra} G_{ar}}{G_{aa}},$$

we obtain for the fourth term in (A.70) that

$$\mathbb{E} \left[ \frac{\gamma^4}{N} \frac{2F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (m - m^{(a)}) \sum_s^{(a)} G_{is} G_{si} \right] = \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}'_{44}] + \mathcal{O}(\Psi^5). \quad (\text{A.74})$$

Finally, similar to (A.50), we get for the last term in (A.70) that

$$\mathbb{E} \left[ \frac{\gamma^2}{N} \frac{2F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X} - \mathfrak{X}^{(a)}) (z + \gamma^2 m - \tau) \sum_s^{(a)} G_{is} G_{si} \right] = \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{32}] + \mathcal{O}(\Psi^5). \quad (\text{A.75})$$

Thus, from (A.70), (A.71), (A.72), (A.73), (A.74) and (A.75), we get

$$\begin{aligned} & \mathbb{E} \left[ \frac{\gamma^2}{N} \frac{2F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (z + \gamma^2 m^{(a)} - \tau) \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} \right] \\ &= \frac{2\gamma^2}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{32}] - \frac{4\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{43}] - \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}'_{44}] \\ & \quad - \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{32}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{A.76})$$

This completes the expansion of the first term in (A.69).

The expansion of the second term in (A.69) was already done in (A.45) and (A.51), which shows that

$$\begin{aligned} & \mathbb{E} \left[ \frac{\gamma^4}{N^2} \frac{4F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \sum_{s,t}^{(a)} G_{is}^{(a)} G_{st}^{(a)} G_{ti}^{(a)} \right] \\ &= \frac{4\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{33}] - \frac{12\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{44}] - \frac{4\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{33}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{A.77})$$

We thus find from (A.69), (A.76) and (A.77) that

$$\begin{aligned} & \mathbb{E} \left[ \frac{2F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \right] \\ &= \frac{2\gamma^2}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{32}] + \frac{4\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{33}] - \frac{4\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{43}] - \frac{12\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{44}] \\ & \quad - \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}'_{44}] - \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{32}] - \frac{4\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{33}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{A.78})$$

**A.5. Expansion of the fifth term in (A.12).** Recall that, by definition,

$$\mathfrak{X} - \mathfrak{X}^{(a)} = \text{Im} \int_{E_1-2}^{E_2-2} \sum_j \frac{\tilde{G}_{ja} \tilde{G}_{aj}}{\tilde{G}_{aa}} d\tilde{y} \quad (\text{A.79})$$

and that  $\mathfrak{X} - \mathfrak{X}^{(a)} = \mathcal{O}(\Psi)$ . We have

$$\begin{aligned} & \frac{2F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X} - \mathfrak{X}^{(a)}) \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \\ &= \frac{2F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \left( \text{Im} \int_{E_1-2}^{E_2-2} \sum_{j,r,m} \tilde{G}_{jr}^{(a)} h_{ra} h_{am} \tilde{G}_{mj}^{(a)} d\tilde{y} \right) \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{A.80})$$

As in (A.37), we take the partial expectation  $\mathbb{E}_a$  to obtain

$$\begin{aligned}
\mathbb{E}_a & \left[ \frac{2F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X} - \mathfrak{X}^{(a)}) \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \right] \\
&= \frac{2F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \frac{\gamma^4}{N^2} \sum_{j,r,s}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jr}^{(a)} \tilde{G}_{rj}^{(a)} d\tilde{y} \right) (z + \gamma^2 m^{(a)} - \tau) G_{is}^{(a)} G_{si}^{(a)} \\
&\quad + \frac{4F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \frac{\gamma^6}{N^3} \sum_{j,r,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jr}^{(a)} \tilde{G}_{rj}^{(a)} d\tilde{y} \right) G_{is}^{(a)} G_{st}^{(a)} G_{ti}^{(a)} \\
&\quad + \frac{4F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \frac{\gamma^4}{N^2} \sum_{j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} \tilde{G}_{tj}^{(a)} d\tilde{y} \right) (z + \gamma^2 m^{(a)} - \tau) G_{is}^{(a)} G_{ti}^{(a)} \\
&\quad + \frac{4F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \frac{\gamma^6}{N^3} \sum_{j,p,q,s}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp}^{(a)} \tilde{G}_{qj}^{(a)} d\tilde{y} \right) G_{is}^{(a)} G_{pq}^{(a)} G_{si}^{(a)} \\
&\quad + \frac{16F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \frac{\gamma^6}{N^3} \sum_{j,p,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp}^{(a)} \tilde{G}_{sj}^{(a)} d\tilde{y} \right) G_{is}^{(a)} G_{pt}^{(a)} G_{ti}^{(a)} + \mathcal{O}(\Psi^5). \tag{A.81}
\end{aligned}$$

We thus have for the fifth term in (A.12) that

$$\begin{aligned}
\mathbb{E} & \left[ \frac{2F''(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^3} (\mathfrak{X} - \mathfrak{X}^{(a)}) (z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau) \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \right] \\
&= \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{32}] + \frac{4\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{33}] \\
&\quad + \frac{4\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} \tilde{G}_{tj}^{(a)} d\tilde{y} \right) (z + \gamma^2 m - \tau) G_{is} G_{ti} \right] \\
&\quad + \frac{4\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,p,q}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp}^{(a)} \tilde{G}_{qj}^{(a)} d\tilde{y} \right) G_{pq} \mathcal{X}_{22} \right] \\
&\quad + \frac{16\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,p,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp}^{(a)} \tilde{G}_{sj}^{(a)} d\tilde{y} \right) G_{is} G_{pt} G_{ti} \right] + \mathcal{O}(\Psi^5). \tag{A.82}
\end{aligned}$$

**A.6. Expansion of the sixth term in (A.12).** We take the partial expectation  $\mathbb{E}_a$  to get

$$\begin{aligned}
\mathbb{E}_a & \left[ \frac{3F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} (z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau)^2 \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \right] \\
&= \frac{3F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \frac{\gamma^2}{N} (z + \gamma^2 \tilde{m}^{(a)} - \tau)^2 \sum_s^{(a)} G_{is}^{(a)} G_{si}^{(a)} + \frac{12F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \frac{\gamma^4}{N^2} (z + \gamma^2 \tilde{m}^{(a)} - \tau) \sum_{s,t}^{(a)} G_{is}^{(a)} G_{st}^{(a)} G_{ti}^{(a)} \\
&\quad + \frac{6F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \frac{\gamma^6}{N^3} \sum_{p,q,s}^{(a)} G_{pq}^{(a)} G_{qp}^{(a)} G_{is}^{(a)} G_{si}^{(a)} + \frac{24F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} \frac{\gamma^6}{N^3} \sum_{p,s,t}^{(a)} G_{ps}^{(a)} G_{tp}^{(a)} G_{is}^{(a)} G_{ti}^{(a)} + \mathcal{O}(\Psi^5). \tag{A.83}
\end{aligned}$$

We thus find for the sixth term in (A.12) that

$$\begin{aligned}
& \mathbb{E} \left[ \frac{3F'(\mathfrak{X}^{(a)})}{(\lambda\gamma v_a - \tau)^4} (z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau)^2 \sum_{s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{ti}^{(a)} \right] \\
&= \frac{3\gamma^2}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{42}] + \frac{12\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{43}] + \frac{6\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}'_{44}] \\
&\quad + \frac{24\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{44}] + \mathcal{O}(\Psi^5). \tag{A.84}
\end{aligned}$$

**A.7. Expansion of (A.12).** We now collect the results (A.57), (A.65), (A.68), (A.78), (A.82) and (A.84). We then find from (A.12) that, for  $a \neq i$ ,

$$\begin{aligned}
& \mathbb{E}[F'(\mathfrak{X})G_{ia}G_{ai}] \\
&= \frac{\gamma^2}{(\lambda\gamma v_a - \tau)^2} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{22}] + \frac{2\gamma^2}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{32}] + \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{33}] \\
&\quad + \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{tj} d\tilde{y} \right) G_{is} G_{ti} \right] \\
&\quad + \frac{3\gamma^2}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{42}] + \frac{6\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{43}] \\
&\quad + \frac{3\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}'_{44}] + \frac{12\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{44}] \\
&\quad + \frac{8\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,r,t,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jr} \tilde{G}_{tj} d\tilde{y} \right) G_{ir} G_{ts} G_{si} \right] \\
&\quad + \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} (z + \gamma^2 \tilde{m} - \tau) \tilde{G}_{js} \tilde{G}_{tj} d\tilde{y} \right) G_{is} G_{ti} \right] \\
&\quad + \frac{4\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,s,t,r} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jr} \tilde{G}_{rs} \tilde{G}_{tj} d\tilde{y} \right) G_{is} G_{ti} \right] \\
&\quad + \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,p,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{st} \tilde{G}_{jp} \tilde{G}_{pj} d\tilde{y} \right) G_{is} G_{ti} \right] \\
&\quad + \frac{4\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F'''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,k,p,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp} \tilde{G}_{sj} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \hat{G}_{kp} \hat{G}_{tk} d\hat{y} \right) G_{is} G_{ti} \right] \\
&\quad + \frac{4\gamma^4}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{tj} d\tilde{y} \right) (z + \gamma^2 m - \tau) G_{is} G_{ti} \right] \\
&\quad + \frac{4\gamma^6}{(\lambda\gamma v_a - \tau)^4} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,p,q} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp} \tilde{G}_{qj} d\tilde{y} \right) G_{pq} \mathcal{X}_{22} \right] + \mathcal{O}(\Psi^5). \tag{A.85}
\end{aligned}$$

We remark that (A.85) is indeed of the form (A.1) claimed at the beginning of this appendix.

## APPENDIX B

In a second step of the proof of Lemma 8.1, we expand in this appendix the second to last term on the right side of (6.11). As before, we always work on the event  $\Xi$  and abbreviate  $\mathcal{O}_\Xi \equiv \mathcal{O}$ . We begin with the expansion

$$\begin{aligned} & \frac{1}{N} \sum_a F'(\mathfrak{X}) G_{ia} G_{bb} G_{ai} \\ &= \frac{1}{N} \sum_a \frac{1}{\lambda \gamma v_b - \tau} F'(\mathfrak{X}) G_{ia} G_{ai} + \frac{1}{N} \sum_a \frac{1}{(\lambda \gamma v_b - \tau)^2} F'(\mathfrak{X}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right) G_{ia} G_{ai} \\ & \quad + \frac{1}{N} \sum_a \frac{1}{(\lambda \gamma v_b - \tau)^3} F'(\mathfrak{X}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right)^2 G_{ia} G_{ai} + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{B.1})$$

Next, we decouple the indices  $a$  and  $b$  in (B.1) so that the index  $b$  appears in the deterministic part only. Since this is already fulfilled in the first term on the right side of (B.1), we keep the first term as it is, i.e.,

$$\mathbb{E} \left[ \frac{1}{N} \sum_a \frac{1}{\lambda \gamma v_b - \tau} F'(\mathfrak{X}) G_{ia} G_{ai} \right] = \frac{1}{\lambda \gamma v_b - \tau} \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{22}]. \quad (\text{B.2})$$

Next, we expand the other two terms on the right side of (B.1) as in the Appendix A.

**B.1. Expansion of the second term in (B.1).** Since

$$F'(\mathfrak{X}) = F'(\mathfrak{X}^{(b)}) + F''(\mathfrak{X}^{(b)})(\mathfrak{X} - \mathfrak{X}^{(b)}) + \mathcal{O}(\Psi^2), \quad (\text{B.3})$$

we have

$$\begin{aligned} & \frac{1}{N} \sum_a \frac{1}{(\lambda \gamma v_b - \tau)^2} F'(\mathfrak{X}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right) G_{ia} G_{ai} \\ &= \frac{1}{N} \sum_a \frac{1}{(\lambda \gamma v_b - \tau)^2} F'(\mathfrak{X}^{(b)}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right) G_{ia}^{(b)} G_{ai}^{(b)} \\ & \quad + \frac{1}{N} \sum_a \frac{1}{(\lambda \gamma v_b - \tau)^2} F'(\mathfrak{X}^{(b)}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right) G_{ia}^{(b)} \frac{G_{ab} G_{bi}}{G_{bb}} \\ & \quad + \frac{1}{N} \sum_a \frac{1}{(\lambda \gamma v_b - \tau)^2} F'(\mathfrak{X}^{(b)}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right) \frac{G_{ib} G_{ba}}{G_{bb}} G_{ai} \\ & \quad + \frac{1}{N} \sum_a \frac{1}{(\lambda \gamma v_b - \tau)^2} F''(\mathfrak{X}^{(b)})(\mathfrak{X} - \mathfrak{X}^{(b)}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right) G_{ia} G_{ai} + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{B.4})$$

Taking the partial expectation  $\mathbb{E}_b$ , we find for the first term in (B.4) that

$$\begin{aligned} & \mathbb{E}_b \left[ \frac{1}{N} \sum_a \frac{1}{(\lambda \gamma v_b - \tau)^2} F'(\mathfrak{X}^{(b)}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right) G_{ia}^{(b)} G_{ai}^{(b)} \right] \\ &= \frac{1}{N} \sum_a \frac{1}{(\lambda \gamma v_b - \tau)^2} F'(\mathfrak{X})(z + \gamma^2 m - \tau) G_{ia} G_{ai} - \frac{1}{N} \sum_a \frac{1}{(\lambda \gamma v_b - \tau)^2} F'(\mathfrak{X}^{(b)})(z + \gamma^2 m^{(b)} - \tau) G_{ia}^{(b)} \frac{G_{ab} G_{bi}}{G_{bb}} \\ & \quad - \frac{1}{N} \sum_a \frac{1}{(\lambda \gamma v_b - \tau)^2} F'(\mathfrak{X}^{(b)})(z + \gamma^2 m^{(b)} - \tau) \frac{G_{ib} G_{ba}}{G_{bb}} G_{ai} - \frac{1}{N} \sum_a \frac{1}{(\lambda \gamma v_b - \tau)^2} F'(\mathfrak{X}^{(b)})(m - m^{(b)}) G_{ia} G_{ai} G_{ai} \\ & \quad - \frac{1}{N} \sum_a \frac{1}{(\lambda \gamma v_b - \tau)^2} F''(\mathfrak{X}^{(b)})(\mathfrak{X} - \mathfrak{X}^{(b)})(z + \gamma^2 m^{(b)} - \tau) G_{ia} G_{ai} + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{B.5})$$

We stop expanding the first term and observe that

$$\mathbb{E} \left[ \frac{1}{N} \sum_a^{(b)} \frac{1}{(\lambda\gamma v_b - \tau)^2} F'(\mathfrak{X})(z + \gamma^2 m - \tau) G_{ia} G_{ai} \right] = \frac{1}{(\lambda\gamma v_b - \tau)^2} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{32}] + \mathcal{O}(\Psi^5). \quad (\text{B.6})$$

All other terms on the right side of (B.5) are of  $\mathcal{O}(\Psi^4)$ . Thus, continuing, we have

$$\mathbb{E} \left[ \frac{1}{N} \sum_a^{(b)} \frac{1}{(\lambda\gamma v_b - \tau)^2} F'(\mathfrak{X}^{(b)})(z + \gamma^2 m^{(b)} - \tau) G_{ia}^{(b)} \frac{G_{ab} G_{bi}}{G_{bb}} \right] = \frac{\gamma^2}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{43}] + \mathcal{O}(\Psi^5), \quad (\text{B.7})$$

which we know from (A.72). Similarly, we get

$$\mathbb{E} \left[ \frac{1}{N} \sum_a^{(b)} \frac{1}{(\lambda\gamma v_b - \tau)^2} F'(\mathfrak{X}^{(b)})(z + \gamma^2 m^{(b)} - \tau) \frac{G_{ib} G_{ba}}{G_{bb}} G_{ai} \right] = \frac{\gamma^2}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{43}] + \mathcal{O}(\Psi^5), \quad (\text{B.8})$$

$$\mathbb{E} \left[ \frac{1}{N} \sum_a^{(b)} \frac{1}{(\lambda\gamma v_b - \tau)^2} F'(\mathfrak{X}^{(b)})(m - m^{(b)}) G_{ia} G_{ai} G_{ai} \right] = \frac{\gamma^2}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}'_{44}] + \mathcal{O}(\Psi^5), \quad (\text{B.9})$$

and, from (A.75),

$$\mathbb{E} \left[ \frac{1}{N} \sum_a^{(b)} \frac{1}{(\lambda\gamma v_b - \tau)^2} F''(\mathfrak{X}^{(b)})(\mathfrak{X} - \mathfrak{X}^{(b)})(z + \gamma^2 m^{(b)} - \tau) G_{ia} G_{ai} \right] = \frac{\gamma^2}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F''(\mathfrak{X})\mathfrak{X}_{22}\mathcal{X}_{32}] + \mathcal{O}(\Psi^5). \quad (\text{B.10})$$

Thus, we find from (B.5), (B.6), (B.7), (B.8), (B.9) and (B.10) that

$$\begin{aligned} & \mathbb{E} \left[ \frac{1}{N} \sum_a^{(b)} \frac{1}{(\lambda\gamma v_b - \tau)^2} F'(\mathfrak{X}^{(b)}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right) G_{ia}^{(b)} G_{ai}^{(b)} \right] \\ &= \frac{1}{(\lambda\gamma v_b - \tau)^2} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{32}] - \frac{2\gamma^2}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{43}] - \frac{\gamma^4}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}'_{44}] \\ & \quad - \frac{\gamma^2}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F''(\mathfrak{X})\mathfrak{X}_{22}\mathcal{X}_{32}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{B.11})$$

In order to expand the second term in (B.4), we first notice that

$$\begin{aligned} & \frac{1}{N} \sum_a^{(b)} \frac{1}{(\lambda\gamma v_b - \tau)^2} F'(\mathfrak{X}^{(b)}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right) G_{ia}^{(b)} \frac{G_{ab} G_{bi}}{G_{bb}} \\ &= \frac{1}{N} \frac{1}{(\lambda\gamma v_b - \tau)^3} F'(\mathfrak{X}^{(b)}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right) \sum_{a,p,q}^{(b)} G_{ia}^{(b)} G_{ap}^{(b)} h_{pb} h_{bq} G_{qi}^{(b)} + \mathcal{O}(\Psi^5). \end{aligned}$$

Then, using (A.52), we find that

$$\begin{aligned} & \mathbb{E} \left[ \frac{1}{N} \sum_a^{(b)} \frac{1}{(\lambda\gamma v_b - \tau)^2} F'(\mathfrak{X}^{(b)}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right) G_{ia}^{(b)} \frac{G_{ab} G_{bi}}{G_{bb}} \right] \\ &= \frac{\gamma^2}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{43}] + \frac{2\gamma^4}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{44}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{B.12})$$

Similarly, we also have

$$\begin{aligned} & \mathbb{E} \left[ \frac{1}{N} \sum_a^{(b)} \frac{1}{(\lambda\gamma v_b - \tau)^2} F'(\mathfrak{X}^{(b)}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right) \frac{G_{ib} G_{ba}}{G_{bb}} G_{ai} \right] \\ &= \frac{\gamma^2}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{43}] + \frac{2\gamma^4}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{44}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{B.13})$$

Finally, for the last term in (B.4), we observe that

$$\begin{aligned} & \frac{1}{N} \sum_a^{(b)} \frac{F''(\mathfrak{X}^{(b)})}{(\lambda\gamma v_b - \tau)^2} (\mathfrak{X} - \mathfrak{X}^{(b)}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right) G_{ia} G_{ai} \\ &= \frac{1}{N} \sum_a^{(b)} \frac{F''(\mathfrak{X}^{(b)})}{(\lambda\gamma v_b - \tau)^3} \left( \text{Im} \int_{E_1-2}^{E_2-2} \sum_{j,r,m}^{(b)} \tilde{G}_{jr}^{(b)} h_{ra} h_{am} \tilde{G}_{mj}^{(b)} d\tilde{y} \right) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right) G_{ia}^{(b)} G_{ai}^{(b)} + \mathcal{O}(\Psi^5). \end{aligned}$$

Thus taking the partial expectation we find

$$\begin{aligned} & \mathbb{E} \left[ \frac{1}{N} \sum_a^{(b)} \frac{F''(\mathfrak{X}^{(b)})}{(\lambda\gamma v_b - \tau)^2} (\mathfrak{X} - \mathfrak{X}^{(b)}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right) G_{ia} G_{ai} \right] \\ &= \frac{\gamma^2}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F''(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{32}] + \frac{2\gamma^4}{(\lambda\gamma v_b - \tau)^3} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,p,q} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp} \tilde{G}_{qj} d\tilde{y} \right) G_{pq} \mathcal{X}_{22} \right] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{B.14})$$

Combining (B.4), (B.11), (B.12), (B.13) and (B.14), we obtain

$$\begin{aligned} & \mathbb{E} \left[ \frac{1}{N} \sum_a^{(b)} \frac{1}{(\lambda\gamma v_b - \tau)^2} F'(\mathfrak{X}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right) G_{ia} G_{ai} \right] \\ &= \frac{1}{(\lambda\gamma v_b - \tau)^2} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{32}] - \frac{\gamma^2}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}'_{44}] + \frac{4\gamma^4}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{44}] \\ &\quad + \frac{2\gamma^4}{(\lambda\gamma v_b - \tau)^3} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,p,q} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp} \tilde{G}_{qj} d\tilde{y} \right) G_{pq} \mathcal{X}_{22} \right] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{B.15})$$

This completes the expansion of the second term in (B.1).

**B.2. Expansion of the third term in (B.1).** We next focus on the third term in (B.1) which is of  $\mathcal{O}(\Psi^4)$ . We have

$$\begin{aligned} & \mathbb{E}_b \left[ \frac{1}{N} \sum_a^{(b)} \frac{1}{(\lambda\gamma v_b - \tau)^3} F'(\mathfrak{X}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right)^2 G_{ia} G_{ai} \right] \\ &= \mathbb{E}_b \left[ \frac{1}{N} \sum_a^{(b)} \frac{1}{(\lambda\gamma v_b - \tau)^3} F'(\mathfrak{X}^{(b)}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right)^2 G_{ia}^{(b)} G_{ai}^{(b)} \right] + \mathcal{O}(\Psi^5) \\ &= \frac{F'(\mathfrak{X}^{(b)})}{(\lambda\gamma v_b - \tau)^3} \frac{1}{N} \sum_a^{(b)} (z + \gamma^2 m^{(b)} - \tau)^2 G_{ia}^{(b)} G_{ai}^{(b)} + \frac{F'(\mathfrak{X}^{(b)})}{(\lambda\gamma v_b - \tau)^3} \frac{\gamma^4}{N} \sum_{a,p,r}^{(b)} G_{pr}^{(b)} G_{rp}^{(b)} G_{ia}^{(b)} G_{ai}^{(b)} + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{B.16})$$

Thus, we get

$$\begin{aligned} & \mathbb{E} \left[ \frac{1}{N} \sum_a^{(b)} \frac{1}{(\lambda\gamma v_b - \tau)^3} F'(\mathfrak{X}) \left( z + \sum_{p,q}^{(b)} h_{bp} G_{pq}^{(b)} h_{qb} - \tau \right)^2 G_{ia} G_{ai} \right] \\ &= \frac{1}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{42}] + \frac{2\gamma^4}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}'_{44}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{B.17})$$



**B.3. Expansion of  $\mathbb{E}[F'(\mathfrak{X})G_{ia}G_{bb}G_{ai}]$ .** From (B.1), (B.2), (B.15) and (B.17), we obtain

$$\begin{aligned} & \mathbb{E} \left[ \frac{1}{N} \sum_a F'(\mathfrak{X}) G_{ia} G_{bb} G_{ai} \right] \\ &= \frac{1}{\lambda\gamma v_b - \tau} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{22}] + \frac{1}{(\lambda\gamma v_b - \tau)^2} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{32}] + \frac{1}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{42}] + \frac{\gamma^4}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}'_{44}] \\ &+ \frac{4\gamma^4}{(\lambda\gamma v_b - \tau)^3} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{44}] + \frac{2\gamma^4}{(\lambda\gamma v_b - \tau)^3} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^2} \sum_{j,p,q} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp} \tilde{G}_{qj} d\tilde{y} \right) G_{pq} \mathcal{X}_{22} \right] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{B.18})$$

**Remark B.1.** Choosing  $F' \equiv 1$  we infer from (6.11), (A.85) and (B.18) that

$$\begin{aligned} \mathbb{E} dG_{ii} &= \left( -\gamma^2 \partial_t(\lambda\gamma) \sum_j \frac{v_j}{(\lambda\gamma v_a - \tau)^2} + \dot{z} + 2\dot{\gamma}\gamma \sum_j \frac{1}{\lambda\gamma v_j - \tau} \right) \mathbb{E}[\mathcal{X}_{22}] dt \\ &+ \left( -2\gamma^2 \partial_t(\lambda\gamma) \sum_j \frac{v_j}{(\lambda\gamma v_a - \tau)^3} + 2\dot{\gamma}\gamma^{-1} \right) \mathbb{E}[\mathcal{X}_{32} + \gamma^2 \mathcal{X}_{33}] dt \\ &- 3\gamma^2 \partial_t(\lambda\gamma) \left( \sum_j \frac{v_j}{(\lambda\gamma v_a - \tau)^4} \right) \mathbb{E}[\mathcal{X}_{42} + 2\gamma^2 \mathcal{X}_{43} + 4\gamma^4 \mathcal{X}_{44} + \gamma^4 \mathcal{X}'_{44}] dt \\ &+ 2\dot{\gamma}\gamma \left( \sum_j \frac{1}{(\lambda\gamma v_a - \tau)^3} \right) \mathbb{E}[\mathcal{X}_{42} + 4\gamma^4 \mathcal{X}_{44} + \gamma^4 \mathcal{X}'_{44}] dt + \mathbb{E}[G_{ii}^2] dt + \mathcal{O}(N^{1/2}\Psi^3) dt, \end{aligned} \quad (\text{B.19})$$

where we used the sum rule

$$\sum_j \frac{1}{(\lambda\gamma v_j - \tau)^2} = \frac{1}{\gamma^2}. \quad (\text{B.20})$$

Note that we have added the term  $G_{ii}$  at the end to account for the case  $a = i$ .

## APPENDIX C

In a third step of the proof of Lemma 8.1, we further simplify in this appendix the right side of (6.11), using (A.85) and (B.18). We always work on the event  $\Xi$  and abbreviate  $\mathcal{O}_{\Xi} \equiv \mathcal{O}$ .

Let

$$X_2 := \frac{1}{N} \sum_i F'(\mathfrak{X}) \mathcal{X}_{22} \quad (\text{C.1})$$

and

$$X_3 := \frac{1}{N} \sum_i \left( F'(\mathfrak{X}) \mathcal{X}_{32} + \gamma^2 F'(\mathfrak{X}) \mathcal{X}_{33} + F''(\mathfrak{X}) \frac{\gamma^2}{N^2} \sum_{j,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{tj} d\tilde{y} \right) G_{is} G_{ti} \right). \quad (\text{C.2})$$

It is obvious that  $X_2 = \mathcal{O}(\Psi^2)$  and  $X_3 = \mathcal{O}(\Psi^3)$ . Furthermore, for  $a \neq i$ , we let  $X_4$  be a random variable not containing  $a$  as a fixed index that satisfies

$$\frac{1}{N} \sum_i \mathbb{E}[F'(\mathfrak{X})G_{ia}G_{ai}] = \frac{\gamma^2}{(\lambda\gamma v_a - \tau)^2} \mathbb{E}X_2 + \frac{2\gamma^2}{(\lambda\gamma v_a - \tau)^3} \mathbb{E}X_3 + \frac{\gamma^2}{(\lambda\gamma v_a - \tau)^4} \mathbb{E}X_4 + \mathcal{O}(\Psi^5) \quad (\text{C.3})$$

and  $X_4 = \mathcal{O}(\Psi^4)$ . We can easily check the existence of such an  $X_4$  from (A.85).

Using the notations in (8.1), we have from (C.3) that, after summing over the index  $a$ ,

$$N\mathbb{E}[X_2] = N\gamma^2 A_2 \mathbb{E}[X_2] + 2N\gamma^2 A_3 \mathbb{E}[X_3] + N\gamma^2 A_4 \mathbb{E}[X_4] + \frac{1}{N} \sum_i \mathbb{E}[F'(\mathfrak{X})G_{ii}^2] + \mathcal{O}(\Psi^2), \quad (\text{C.4})$$

which also implies

$$\mathbb{E}[X_4] = -\frac{2A_3}{A_4} \mathbb{E}[X_3] - \frac{1}{\gamma^2 A_4 N^2} \sum_i \mathbb{E}[F'(\mathfrak{X})G_{ii}^2] + \mathcal{O}(\Psi^5), \quad (\text{C.5})$$

where we used that  $A_2 = \gamma^{-2}$ .

**C.1. Simplification of (A.85).** We next consider the first term (6.11), where we have an additional factor  $v_a$  in the summand. Since  $v_a$  is considered fix, the addition of such a factor does not change the expansion results we have obtained so far. We hence get

$$\begin{aligned}
& \frac{1}{N} \sum_{i,a} v_a \mathbb{E} [F'(\mathfrak{X}) G_{ia} G_{ai}] \\
&= \sum_a \frac{\gamma^2 v_a}{(\lambda \gamma v_a - \tau)^2} \mathbb{E} X_2 + \sum_a \frac{2\gamma^2 v_a}{(\lambda \gamma v_a - \tau)^3} \mathbb{E} X_3 + \sum_a \frac{\gamma^2 v_a}{(\lambda \gamma v_a - \tau)^4} \mathbb{E} X_4 + \frac{1}{N} \sum_i \mathbb{E} [v_i F'(\mathfrak{X}) G_{ii}^2] \\
&= N\gamma^2 A'_2 \mathbb{E} X_2 + 2N\gamma^2 A'_3 \mathbb{E} X_3 + N\gamma^2 A'_4 \mathbb{E} X_4 + \frac{1}{N} \sum_i \mathbb{E} [v_i F'(\mathfrak{X}) G_{ii}^2] + \mathcal{O}(\Psi^2) \\
&= N\gamma^2 A'_2 \mathbb{E} X_2 + 2N\gamma^2 \left( A'_3 - \frac{A_3 A'_4}{A_4} \right) \mathbb{E} X_3 - \frac{A'_4}{A_4} \frac{1}{N} \sum_i \mathbb{E} [F'(\mathfrak{X}) G_{ii}^2] + \frac{1}{N} \sum_i \mathbb{E} [v_i F'(\mathfrak{X}) G_{ii}^2] + \mathcal{O}(\Psi^2). \quad (\text{C.6})
\end{aligned}$$

The last two terms on the right side of (C.6) can be further expanded using

$$F'(\mathfrak{X}) G_{ii}^2 = \frac{1}{(\lambda \gamma v_i - \tau)^2} F'(\mathfrak{X}) + \frac{2}{(\lambda \gamma v_i - \tau)^3} \left( z + \sum_{p,q}^{(i)} h_{ip} G_{pq}^{(i)} h_{qi} - \tau \right) F'(\mathfrak{X}) + \mathcal{O}(\Psi^2).$$

We further observe that

$$\begin{aligned}
\mathbb{E}_i \left[ \left( z + \sum_{p,q}^{(i)} h_{ip} G_{pq}^{(i)} h_{qi} - \tau \right) F'(\mathfrak{X}) \right] &= \mathbb{E}_i \left[ \left( z + \sum_{p,q}^{(i)} h_{ip} G_{pq}^{(i)} h_{qi} - \tau \right) F'(\mathfrak{X}^{(i)}) \right] + \mathcal{O}(\Psi^2) \\
&= (z + \gamma^2 m^{(i)} - \tau) F'(\mathfrak{X}^{(i)}) + \mathcal{O}(\Psi^2) \\
&= (z + \gamma^2 m - \tau) F'(\mathfrak{X}) + \mathcal{O}(\Psi^2),
\end{aligned}$$

and we hence obtain

$$\mathbb{E} [F'(\mathfrak{X}) G_{ii}^2] = \frac{1}{(\lambda \gamma v_i - \tau)^2} \mathbb{E} [F'(\mathfrak{X})] + \frac{2}{(\lambda \gamma v_i - \tau)^3} \mathbb{E} [(z + \gamma^2 m - \tau) F'(\mathfrak{X})] + \mathcal{O}(\Psi^2).$$

We thus have

$$\frac{1}{N} \sum_i \mathbb{E} [F'(\mathfrak{X}) G_{ii}^2] = A_2 \mathbb{E} [F'(\mathfrak{X})] + 2A_3 \mathbb{E} [(z + \gamma^2 m - \tau) F'(\mathfrak{X})] + \mathcal{O}(\Psi^2) \quad (\text{C.7})$$

and, similarly,

$$\frac{1}{N} \sum_i \mathbb{E} [v_i F'(\mathfrak{X}) G_{ii}^2] = A'_2 \mathbb{E} [F'(\mathfrak{X})] + 2A'_3 \mathbb{E} [(z + \gamma^2 m - \tau) F'(\mathfrak{X})] + \mathcal{O}(\Psi^2). \quad (\text{C.8})$$

Putting (C.7) and (C.8) back into (C.6), we find

$$\begin{aligned}
\frac{1}{N} \sum_{i,a} v_a \mathbb{E} [F'(\mathfrak{X}) G_{ia} G_{ai}] &= N\gamma^2 A'_2 \mathbb{E} [X_2] + 2N\gamma^2 \left( A'_3 - \frac{A_3 A'_4}{A_4} \right) \mathbb{E} [X_3] + \left( A'_2 - \frac{A_2 A'_4}{A_4} \right) \mathbb{E} [F'(\mathfrak{X})] \\
&\quad + \left( A'_3 - \frac{A_3 A'_4}{A_4} \right) \mathbb{E} [(z + \gamma^2 m - \tau) F'(\mathfrak{X})] + \mathcal{O}(\Psi^2). \quad (\text{C.9})
\end{aligned}$$

We now go back to the other terms in (6.11). We have from (B.18) that

$$\begin{aligned}
& \frac{1}{N^2} \sum_{i,a,b} (\mathbb{E} [F'(\mathfrak{X}) G_{ia} G_{ab} G_{bi}] + \mathbb{E} [F'(\mathfrak{X}) G_{ia} G_{bb} G_{ai}]) \\
&\quad + \frac{1}{N^2} \sum_{i,a,b,j} \mathbb{E} \left[ F''(\mathfrak{X}) \left( \text{Im} \int_{E_1-2}^{E_2-2} G'_{ja} G'_{bj} d\tilde{y} \right) G_{ia} G_{bi} \right] \\
&= N A_1 \mathbb{E} [X_2] + N A_2 \mathbb{E} [X_3] + A_3 \sum_i (\mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{42}] + \gamma^4 \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}'_{44}] + 4\gamma^4 \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{44}]) \\
&\quad + 2\gamma^4 A_3 \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^3} \sum_{i,j,p,q,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp} \tilde{G}_{qj} d\tilde{y} \right) G_{pq} G_{is} G_{si} \right] + \mathcal{O}(\Psi^2). \quad (\text{C.10})
\end{aligned}$$

We note that the right side of (C.10) contains several terms of  $\mathcal{O}(\Psi)$  that are not directly related to  $X_4$ . In order to compare those terms with the terms in (C.9), we expand the terms in  $X_3$  to find more “optical theorems”.

**C.2. Optical theorem from  $\mathcal{X}_{32}$ .** For  $F'(\mathfrak{X})\mathcal{X}_{32}$ , we consider

$$F'(\mathfrak{X})\mathcal{X}_{32} = \frac{F'(\mathfrak{X})}{N}(z + \gamma^2 m - \tau) \sum_a G_{ia} G_{ai} = \frac{F'(\mathfrak{X})}{N}(z + \gamma^2 m - \tau) \left( G_{ii}^2 + \sum_a^{(i)} G_{ia} G_{ai} \right) \quad (\text{C.11})$$

and expand each term on the right side. We can easily see that the first term of (C.11) becomes

$$\frac{F'(\mathfrak{X})}{N}(z + \gamma^2 m - \tau) G_{ii}^2 = \frac{1}{(\lambda \gamma v_i - \tau)^2} \frac{F'(\mathfrak{X})}{N}(z + \gamma^2 m - \tau) + \mathcal{O}(\Psi^2). \quad (\text{C.12})$$

We next expand the second term of (C.11) to find that, for  $a \neq i$ ,

$$\begin{aligned} & F'(\mathfrak{X})(z + \gamma^2 m - \tau) G_{ia} G_{ai} \\ &= \frac{1}{(\lambda \gamma v_a - \tau)^2} F'(\mathfrak{X})(z + \gamma^2 m - \tau) \sum_{p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qi}^{(a)} \\ &+ \frac{2}{(\lambda \gamma v_a - \tau)^3} F'(\mathfrak{X})(z + \gamma^2 m - \tau) \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) \sum_{p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qi}^{(a)} + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{C.13})$$

Expanding further the first term in (C.13), we get

$$\begin{aligned} & \frac{1}{(\lambda \gamma v_a - \tau)^2} F'(\mathfrak{X})(z + \gamma^2 m - \tau) \sum_{p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qi}^{(a)} \\ &= \frac{1}{(\lambda \gamma v_a - \tau)^2} F'(\mathfrak{X}^{(a)})(z + \gamma^2 m^{(a)} - \tau) \sum_{p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qi}^{(a)} \\ &+ \frac{\gamma^2}{(\lambda \gamma v_i - \tau)^2} F'(\mathfrak{X}^{(a)})(m - m^{(a)}) \sum_{p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qi}^{(a)} \\ &+ \frac{1}{(\lambda \gamma v_i - \tau)^2} F''(\mathfrak{X}^{(a)})(\mathfrak{X} - \mathfrak{X}^{(a)})(z + \gamma^2 m - \tau) \sum_{p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qi}^{(a)} + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{C.14})$$

We then take the partial expectation  $\mathbb{E}_a$  for each term in (C.14). The first term yields

$$\begin{aligned} & \mathbb{E}_a \left[ \frac{1}{(\lambda \gamma v_a - \tau)^2} F'(\mathfrak{X}^{(a)})(z + \gamma^2 m^{(a)} - \tau) \sum_{p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qi}^{(a)} \right] \\ &= \frac{\gamma^2}{N} \frac{1}{(\lambda \gamma v_a - \tau)^2} F'(\mathfrak{X}^{(a)})(z + \gamma^2 m^{(a)} - \tau) \sum_p^{(a)} G_{ip}^{(a)} G_{pi}^{(a)}. \end{aligned} \quad (\text{C.15})$$

Thus, taking the full expectation we obtain with (A.76) that

$$\begin{aligned} & \mathbb{E} \left[ \frac{1}{(\lambda \gamma v_a - \tau)^2} F'(\mathfrak{X}^{(a)})(z + \gamma^2 m^{(a)} - \tau) \sum_{p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qi}^{(a)} \right] \\ &= \frac{\gamma^2}{(\lambda \gamma v_a - \tau)^2} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{32}] - \frac{2\gamma^4}{(\lambda \gamma v_a - \tau)^3} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{43}] - \frac{\gamma^6}{(\lambda \gamma v_a - \tau)^3} \mathbb{E}[F'(\mathfrak{X})\mathcal{X}'_{44}] \\ &- \frac{\gamma^4}{(\lambda \gamma v_a - \tau)^3} \mathbb{E}[F''(\mathfrak{X})\mathfrak{X}_{22}\mathcal{X}_{32}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{C.16})$$

The second term in (C.14) can be expanded similarly, using the relation

$$m - m^{(a)} = \frac{1}{N} \sum_r \frac{G_{ra} G_{ar}}{G_{aa}}.$$

We then have

$$\begin{aligned} & \mathbb{E} \left[ \frac{\gamma^2}{(\lambda\gamma v_i - \tau)^2} F'(\mathfrak{X}^{(a)})(m - m^{(a)}) \sum_{p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qi}^{(a)} \right] \\ &= \frac{\gamma^6}{(\lambda\gamma v_a - \tau)^3} \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}'_{44}] + \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^3} \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{44}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{C.17})$$

Finally, we have for the third term in (C.14), from the relation

$$\mathfrak{X} - \mathfrak{X}^{(a)} = \text{Im} \int_{E_1-2}^{E_2-2} \sum_j \frac{\tilde{G}_{ja} \tilde{G}_{aj}}{\tilde{G}_{aa}} d\tilde{y},$$

that

$$\begin{aligned} & \mathbb{E} \left[ \frac{1}{(\lambda\gamma v_i - \tau)^2} F''(\mathfrak{X}^{(a)})(\mathfrak{X} - \mathfrak{X}^{(a)})(z + \gamma^2 m - \tau) \sum_{p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qi}^{(a)} \right] \\ &= \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E} [F'(\mathfrak{X}) \mathfrak{X}_{22} \mathcal{X}_{32}] \\ &+ \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^2} \sum_{r,p,q} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{rp} \tilde{G}_{qr} d\tilde{y} \right) (z + \gamma^2 m - \tau) G_{ip} G_{qi} \right] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{C.18})$$

Thus, from (C.14), (C.16), (C.17) and (C.18), we get

$$\begin{aligned} & \mathbb{E} \left[ \frac{1}{(\lambda\gamma v_a - \tau)^2} F'(\mathfrak{X})(z + \gamma^2 m - \tau) \sum_{p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qi}^{(a)} \right] \\ &= \frac{\gamma^2}{(\lambda\gamma v_a - \tau)^2} \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{32}] - \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{43}] + \frac{2\gamma^6}{(\lambda\gamma v_a - \tau)^3} \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{44}] \\ &+ \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^2} \sum_{r,p,q} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{rp} \tilde{G}_{qr} d\tilde{y} \right) (z + \gamma^2 m - \tau) G_{ip} G_{qi} \right] + \mathcal{O}(\Psi^5), \end{aligned} \quad (\text{C.19})$$

which completes the expansion of the first term in (C.13). The second term in (C.13) is of  $\mathcal{O}(\Psi^2)$  and we observe that

$$\begin{aligned} & \mathbb{E} \left[ \frac{2}{(\lambda\gamma v_a - \tau)^3} F'(\mathfrak{X})(z + \gamma^2 m - \tau) \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) \sum_{p,q}^{(a)} G_{ip}^{(a)} h_{pa} h_{aq} G_{qi}^{(a)} \right] \\ &= \frac{2\gamma^2}{(\lambda\gamma v_a - \tau)^3} \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{42}] + \frac{4\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{43}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{C.20})$$

Combining (C.11), (C.12), (C.13), (C.19) and (C.20), we conclude that

$$\begin{aligned} \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{32}] &= \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{32}] + 2A_3(\gamma^2 \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{42}] + \gamma^4 \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{43}] + \gamma^6 \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{44}]) \\ &+ 2A_3\gamma^4 \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^2} \sum_{r,p,q} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{rp} \tilde{G}_{qr} d\tilde{y} \right) (z + \gamma^2 m - \tau) G_{ip} G_{qi} \right] \\ &+ \frac{1}{(\lambda\gamma v_i - \tau)^2} \frac{1}{N} \mathbb{E} [F'(\mathfrak{X})(z + \gamma^2 m - \tau)] + \mathcal{O}(\Psi^5), \end{aligned} \quad (\text{C.21})$$

which implies the first “optical theorem” of this appendix,

$$\begin{aligned} & \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{42}] + \gamma^2 \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{43}] + \gamma^4 \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{44}] \\ &+ \gamma^2 \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^2} \sum_{r,p,q} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{rp} \tilde{G}_{qr} d\tilde{y} \right) (z + \gamma^2 m - \tau) G_{ip} G_{qi} \right] \\ &= -\frac{1}{2\gamma^2 A_3} \frac{1}{(\lambda\gamma v_i - \tau)^2} \frac{1}{N} \mathbb{E} [F'(\mathfrak{X})(z + \gamma^2 m - \tau)] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{C.22})$$

**C.3. Optical theorem from  $\mathcal{X}_{33}$ .** We perform a similar expansion for  $F'(\mathfrak{X})\mathcal{X}_{33}$ . We first notice that

$$\begin{aligned} F'(\mathfrak{X})\frac{1}{N}\sum_b G_{ia}G_{ab}G_{bi} &= \frac{1}{(\lambda\gamma v_a - \tau)^2}\frac{F'(\mathfrak{X})}{N}\sum_{b,s,t}^{(a)} G_{is}^{(a)}h_{sa}h_{at}G_{tb}^{(a)}G_{bi} \\ &\quad + \frac{2}{(\lambda\gamma v_a - \tau)^3}\frac{F'(\mathfrak{X})}{N}\left(z + \sum_{p,q}^{(a)} h_{ap}G_{pq}^{(a)}h_{qa} - \tau\right)\sum_{b,s,t}^{(a)} G_{is}^{(a)}h_{sa}h_{at}G_{tb}^{(a)}G_{bi} + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{C.23})$$

The first term in (C.23) can be written as

$$\begin{aligned} \frac{1}{(\lambda\gamma v_a - \tau)^2}\frac{F'(\mathfrak{X})}{N}\sum_{b,s,t}^{(a)} G_{is}^{(a)}h_{sa}h_{at}G_{tb}^{(a)}G_{bi} &= \frac{1}{(\lambda\gamma v_a - \tau)^2}\frac{F'(\mathfrak{X}^{(a)})}{N}\sum_{b,s,t}^{(a)} G_{is}^{(a)}h_{sa}h_{at}G_{tb}^{(a)}G_{bi}^{(a)} + \frac{1}{(\lambda\gamma v_a - \tau)^2}\frac{F'(\mathfrak{X}^{(a)})}{N}\sum_{b,s,t}^{(a)} G_{is}^{(a)}h_{sa}h_{at}G_{tb}^{(a)}\frac{G_{ba}G_{ai}}{G_{aa}} \\ &\quad + \frac{1}{(\lambda\gamma v_a - \tau)^2}\frac{F''(\mathfrak{X}^{(a)})}{N}(\mathfrak{X} - \mathfrak{X}^{(a)})\sum_{b,s,t}^{(a)} G_{is}^{(a)}h_{sa}h_{at}G_{tb}^{(a)}G_{bi} + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{C.24})$$

The expectation of the first term in (C.24) has already been computed in (A.51), which gives us

$$\begin{aligned} \mathbb{E}\left[\frac{1}{(\lambda\gamma v_a - \tau)^2}\frac{F'(\mathfrak{X}^{(a)})}{N}\sum_{b,s,t}^{(a)} G_{is}^{(a)}h_{sa}h_{at}G_{tb}^{(a)}G_{bi}^{(a)}\right] &= \frac{\gamma^2}{(\lambda\gamma v_a - \tau)^2}\mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{33}] - \frac{3\gamma^4}{(\lambda\gamma v_a - \tau)^3}\mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{44}] - \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^3}\mathbb{E}[F''(\mathfrak{X})\mathfrak{X}_{22}\mathcal{X}_{33}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{C.25})$$

The expansion of the second term in (C.24) is similar to the one in (A.54) and we get

$$\begin{aligned} \mathbb{E}\left[\frac{1}{(\lambda\gamma v_a - \tau)^2}\frac{F'(\mathfrak{X}^{(a)})}{N}\sum_{b,s,t}^{(a)} G_{is}^{(a)}h_{sa}h_{at}G_{tb}^{(a)}\frac{G_{ba}G_{ai}}{G_{aa}}\right] &= \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^3}\mathbb{E}[F'(\mathfrak{X})\mathcal{X}'_{44}] + \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^3}\mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{44}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{C.26})$$

The third term in (C.24) yields, using (A.37),

$$\begin{aligned} \mathbb{E}\left[\frac{1}{(\lambda\gamma v_a - \tau)^2}\frac{F''(\mathfrak{X}^{(a)})}{N}(\mathfrak{X} - \mathfrak{X}^{(a)})\sum_{b,s,t}^{(a)} G_{is}^{(a)}h_{sa}h_{at}G_{tb}^{(a)}G_{bi}\right] &= \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^3}\mathbb{E}[F''(\mathfrak{X})\mathfrak{X}_{22}\mathcal{X}_{33}] + \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^3}\mathbb{E}\left[F''(\mathfrak{X})\frac{1}{N^3}\sum_{j,r,t,s}\left(\text{Im}\int_{E_1-2}^{E_2-2}\tilde{G}_{jr}\tilde{G}_{tj}d\tilde{y}\right)G_{ir}G_{ts}G_{si}\right] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{C.27})$$

Thus, we obtain from (C.24), (C.25), (C.26) and (C.27) that

$$\begin{aligned} \mathbb{E}\left[\frac{1}{(\lambda\gamma v_a - \tau)^2}\frac{F'(\mathfrak{X})}{N}\sum_{b,s,t}^{(a)} G_{is}^{(a)}h_{sa}h_{at}G_{tb}^{(a)}G_{bi}\right] &= \frac{\gamma^2}{(\lambda\gamma v_a - \tau)^2}\mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{33}] - \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^3}\mathbb{E}[F'(\mathfrak{X})\mathcal{X}_{44}] + \frac{\gamma^4}{(\lambda\gamma v_a - \tau)^3}\mathbb{E}[F'(\mathfrak{X})\mathcal{X}'_{44}] \\ &\quad + \frac{2\gamma^4}{(\lambda\gamma v_a - \tau)^3}\mathbb{E}\left[F''(\mathfrak{X})\frac{1}{N^3}\sum_{j,r,t,s}\left(\text{Im}\int_{E_1-2}^{E_2-2}\tilde{G}_{jr}\tilde{G}_{tj}d\tilde{y}\right)G_{ir}G_{ts}G_{si}\right] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{C.28})$$

The second term in (C.23) can be expanded as in (A.52) and we find that

$$\begin{aligned} & \mathbb{E} \left[ \frac{2}{(\lambda\gamma v_a - \tau)^3} \frac{F'(\mathfrak{X})}{N} \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) \sum_{b,s,t}^{(a)} G_{is}^{(a)} h_{sa} h_{at} G_{tb}^{(a)} G_{bi} \right] \\ &= \frac{2\gamma^2}{(\lambda\gamma v_a - \tau)^3} \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{43}] + \frac{4\gamma^4}{(\lambda\gamma v_a - \tau)^3} \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{44}] + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{C.29})$$

We thus now have from (C.23), (C.28) and (C.29) that

$$\begin{aligned} \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{33}] &= \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{33}] + \gamma^2 A_3 (2\mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{43}] + \gamma^4 \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}'_{44}] + 3\gamma^4 \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{44}]) \\ &+ 2\gamma^4 A_3 \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,r,t,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jr} \tilde{G}_{ts} d\tilde{y} \right) G_{ir} G_{ts} G_{si} \right] + \mathcal{O}(\Psi^5), \end{aligned}$$

which implies the second “optical theorem” of this appendix,

$$\begin{aligned} & 2\mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{43}] + \gamma^4 \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}'_{44}] + 3\gamma^4 \mathbb{E} [F'(\mathfrak{X}) \mathcal{X}_{44}] + 2\gamma^2 \mathbb{E} \left[ F''(\mathfrak{X}) \frac{1}{N^3} \sum_{j,r,t,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jr} \tilde{G}_{ts} d\tilde{y} \right) G_{ir} G_{ts} G_{si} \right] \\ &= \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{C.30})$$

**C.4. Optical theorem from the other term in  $X_3$ .** We next expand the term

$$\begin{aligned} & \frac{1}{N} \sum_{b,j} F''(\mathfrak{X}) \left( \text{Im} \int_{E_1-2}^{E_2-2} G'_{ja} G'_{bj} d\tilde{y} \right) G_{ia} G_{bi} \\ &= \frac{1}{(\lambda\gamma v_j - \tau)^2} \frac{F''(\mathfrak{X})}{N} \sum_{b,j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} h_{sa} \tilde{G}_{bj} d\tilde{y} \right) G_{it}^{(a)} h_{ta} G_{bi} \\ &+ \frac{1}{(\lambda\gamma v_j - \tau)^3} \frac{F''(\mathfrak{X})}{N} \sum_{b,j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \left( z + \sum_{p,q}^{(a)} h_{ap} \tilde{G}_{pq}^{(a)} h_{qa} - \tau \right) \tilde{G}_{js}^{(a)} h_{sa} \tilde{G}_{bj} d\tilde{y} \right) G_{it}^{(a)} h_{ta} G_{bi} \\ &+ \frac{1}{(\lambda\gamma v_j - \tau)^3} \frac{F''(\mathfrak{X})}{N} \sum_{b,j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} h_{sa} \tilde{G}_{bj} d\tilde{y} \right) \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) G_{it}^{(a)} h_{ta} G_{bi} + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{C.31})$$

We further expand the first term in (C.31) to find

$$\begin{aligned} & \frac{1}{(\lambda\gamma v_j - \tau)^2} \frac{F''(\mathfrak{X})}{N} \sum_{b,j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} h_{sa} \tilde{G}_{bj} d\tilde{y} \right) G_{it}^{(a)} h_{ta} G_{bi} \\ &= \frac{1}{(\lambda\gamma v_j - \tau)^2} \frac{F''(\mathfrak{X}^{(a)})}{N} \sum_{b,j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} h_{sa} \tilde{G}_{bj}^{(a)} d\tilde{y} \right) G_{it}^{(a)} h_{ta} G_{bi}^{(a)} \\ &+ \frac{1}{(\lambda\gamma v_j - \tau)^2} \frac{F''(\mathfrak{X}^{(a)})}{N} \sum_{b,j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} h_{sa} \tilde{G}_{bj}^{(a)} d\tilde{y} \right) G_{it}^{(a)} h_{ta} \frac{G_{ba} G_{ai}}{G_{aa}} \\ &+ \frac{1}{(\lambda\gamma v_j - \tau)^2} \frac{F''(\mathfrak{X}^{(a)})}{N} \sum_{b,j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} h_{sa} \frac{\tilde{G}_{ba} G_{aj}}{G_{aa}} d\tilde{y} \right) G_{it}^{(a)} h_{ta} G_{bi}^{(a)} \\ &+ \frac{1}{(\lambda\gamma v_j - \tau)^2} \frac{F'''(\mathfrak{X}^{(a)})}{N} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_{b,j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} h_{sa} \tilde{G}_{bj}^{(a)} d\tilde{y} \right) G_{it}^{(a)} h_{ta} G_{bi}^{(a)} + \mathcal{O}(\Psi^5). \end{aligned} \quad (\text{C.32})$$

Taking the partial expectation  $\mathbb{E}_a$ , we obtain

$$\begin{aligned}
\mathbb{E}_a & \left[ \frac{1}{(\lambda\gamma v_j - \tau)^2} \frac{F''(\mathfrak{X}^{(a)})}{N} \sum_{b,j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} h_{sa} \tilde{G}_{bj}^{(a)} d\tilde{y} \right) G_{it}^{(a)} h_{ta} G_{bi}^{(a)} \right] \\
&= \frac{\gamma^2}{(\lambda\gamma v_j - \tau)^2} \frac{F''(\mathfrak{X}^{(a)})}{N^2} \sum_{b,j,s}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} \tilde{G}_{bj}^{(a)} d\tilde{y} \right) G_{is}^{(a)} G_{bi}^{(a)} \\
&= \frac{\gamma^2}{(\lambda\gamma v_j - \tau)^2} \frac{F''(\mathfrak{X})}{N^2} \sum_{b,j,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{is} G_{bi} \\
&\quad - \frac{\gamma^2}{(\lambda\gamma v_j - \tau)^2} \frac{F''(\mathfrak{X}^{(a)})}{N^2} \sum_{b,j,s}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} \tilde{G}_{bj}^{(a)} d\tilde{y} \right) G_{is}^{(a)} \frac{G_{ba} G_{ai}}{G_{aa}} \\
&\quad - \frac{\gamma^2}{(\lambda\gamma v_j - \tau)^2} \frac{F''(\mathfrak{X}^{(a)})}{N^2} \sum_{b,j,s}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} \tilde{G}_{bj}^{(a)} d\tilde{y} \right) \frac{G_{ia} G_{as}}{G_{aa}} G_{bi} \\
&\quad - \frac{\gamma^2}{(\lambda\gamma v_j - \tau)^2} \frac{F''(\mathfrak{X}^{(a)})}{N^2} \sum_{b,j,s}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} \frac{\tilde{G}_{ba} \tilde{G}_{aj}}{\tilde{G}_{aa}} d\tilde{y} \right) G_{is} G_{bi} \\
&\quad - \frac{\gamma^2}{(\lambda\gamma v_j - \tau)^2} \frac{F''(\mathfrak{X}^{(a)})}{N^2} \sum_{b,j,s}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \frac{\tilde{G}_{ja} \tilde{G}_{as}}{\tilde{G}_{aa}} \tilde{G}_{bj} d\tilde{y} \right) G_{is} G_{bi} \\
&\quad - \frac{\gamma^2}{(\lambda\gamma v_j - \tau)^2} \frac{F'''(\mathfrak{X}^{(a)})}{N^2} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_{b,j,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{is} G_{bi} + \mathcal{O}(\Psi^5). \tag{C.33}
\end{aligned}$$

Taking the full expectation, we find for the first term in (C.32) that

$$\begin{aligned}
\mathbb{E} & \left[ \frac{1}{(\lambda\gamma v_j - \tau)^2} \frac{F''(\mathfrak{X}^{(a)})}{N} \sum_{b,j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} h_{sa} \tilde{G}_{bj}^{(a)} d\tilde{y} \right) G_{it}^{(a)} h_{ta} G_{bi}^{(a)} \right] \\
&= \frac{\gamma^2}{(\lambda\gamma v_j - \tau)^2} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^2} \sum_{b,j,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{is} G_{bi} \right] \\
&\quad - \frac{2\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{b,j,s,p} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{is} G_{bp} G_{pi} \right] \\
&\quad - \frac{2\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{b,j,s,p} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bp} \tilde{G}_{pj} d\tilde{y} \right) G_{is} G_{bi} \right] \\
&\quad - \frac{\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F'''(\mathfrak{X})}{N^2} \mathfrak{X}_{22} \sum_{b,j,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{is} G_{bi} \right] + \mathcal{O}(\Psi^5). \tag{C.34}
\end{aligned}$$

We expand the other terms in (C.32). The second term becomes

$$\begin{aligned}
\mathbb{E} & \left[ \frac{1}{(\lambda\gamma v_j - \tau)^2} \frac{F''(\mathfrak{X}^{(a)})}{N} \sum_{b,j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} h_{sa} \tilde{G}_{bj}^{(a)} d\tilde{y} \right) G_{it}^{(a)} h_{ta} \frac{G_{ba} G_{ai}}{G_{aa}} \right] \\
&= \frac{2\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{b,j,s,p} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{is} G_{bp} G_{pi} \right] \\
&\quad + \frac{\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{b,j,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{it} G_{bs} G_{ti} \right] + \mathcal{O}(\Psi^5). \tag{C.35}
\end{aligned}$$



Similarly,

$$\begin{aligned}
& \mathbb{E} \left[ \frac{1}{(\lambda\gamma v_j - \tau)^2} \frac{F''(\mathfrak{X}^{(a)})}{N} \sum_{b,j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} h_{sa} \frac{\tilde{G}_{ba} G_{aj}}{G_{aa}} d\tilde{y} \right) G_{it}^{(a)} h_{ta} G_{bi}^{(a)} \right] \\
&= \frac{2\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{b,j,s,p} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bp} \tilde{G}_{pj} d\tilde{y} \right) G_{is} G_{bi} \right] \\
&\quad + \frac{\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{b,j,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bt} \tilde{G}_{sj} d\tilde{y} \right) G_{it} G_{bi} \right] + \mathcal{O}(\Psi^5). \tag{C.36}
\end{aligned}$$

Finally, the fourth term in (C.32) yields

$$\begin{aligned}
& \mathbb{E} \left[ \frac{1}{(\lambda\gamma v_j - \tau)^2} \frac{F'''(\mathfrak{X}^{(a)})}{N} (\mathfrak{X} - \mathfrak{X}^{(a)}) \sum_{b,j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} h_{sa} \tilde{G}_{bj}^{(a)} d\tilde{y} \right) G_{it}^{(a)} h_{ta} G_{bi}^{(a)} \right] \\
&= \frac{\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F'''(\mathfrak{X})}{N^2} \mathfrak{X}_{22} \sum_{b,j,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{is} G_{bi} \right] \\
&\quad + \frac{2\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F'''(\mathfrak{X})}{N^2} \sum_{b,j,k,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \hat{G}_{ks} \hat{G}_{tk} d\hat{y} \right) G_{is} G_{bi} \right] + \mathcal{O}(\Psi^5). \tag{C.37}
\end{aligned}$$

Thus, from (C.32), (C.34), (C.35), (C.36) and (C.37), we obtain

$$\begin{aligned}
& \mathbb{E} \left[ \frac{1}{(\lambda\gamma v_j - \tau)^2} \frac{F''(\mathfrak{X})}{N} \sum_{b,j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} h_{sa} \tilde{G}_{bj} d\tilde{y} \right) G_{it}^{(a)} h_{ta} G_{bi} \right] \\
&= \frac{\gamma^2}{(\lambda\gamma v_j - \tau)^2} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^2} \sum_{b,j,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{is} G_{bi} \right] \\
&\quad + \frac{\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{b,j,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{it} G_{bs} G_{ti} \right] \\
&\quad + \frac{\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{b,j,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bt} \tilde{G}_{sj} d\tilde{y} \right) G_{it} G_{bi} \right] \\
&\quad + \frac{2\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F'''(\mathfrak{X})}{N^2} \sum_{b,j,k,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \hat{G}_{ks} \hat{G}_{tk} d\hat{y} \right) G_{it} G_{bi} \right] + \mathcal{O}(\Psi^5). \tag{C.38}
\end{aligned}$$

The other terms in (C.31) are of  $\mathcal{O}(\Psi^4)$  and we observe that

$$\begin{aligned}
& \mathbb{E} \left[ \frac{1}{(\lambda\gamma v_j - \tau)^3} \frac{F''(\mathfrak{X})}{N} \sum_{b,j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \left( z + \sum_{p,q}^{(a)} h_{ap} \tilde{G}_{pq}^{(a)} h_{qa} - \tau \right) \tilde{G}_{js}^{(a)} h_{sa} \tilde{G}_{bj} d\tilde{y} \right) G_{it}^{(a)} h_{ta} G_{bi} \right] \\
&= \frac{\gamma^2}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^2} \sum_{b,j,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} (z + \gamma^2 \tilde{m} - \tau) \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{is} G_{bi} \right] \\
&\quad + \frac{2\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{b,j,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{st} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{it} G_{bi} \right] + \mathcal{O}(\Psi^5) \tag{C.39}
\end{aligned}$$

and, similarly,

$$\begin{aligned}
& \mathbb{E} \left[ \frac{1}{(\lambda\gamma v_j - \tau)^3} \frac{F''(\mathfrak{X})}{N} \sum_{b,j,s,t}^{(a)} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js}^{(a)} h_{sa} \tilde{G}_{bj} d\tilde{y} \right) \left( z + \sum_{p,q}^{(a)} h_{ap} G_{pq}^{(a)} h_{qa} - \tau \right) G_{it}^{(a)} h_{ta} G_{bi} \right] \\
&= \frac{\gamma^2}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^2} \sum_{b,j,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) (z + \gamma^2 m - \tau) G_{is} G_{bi} \right] \\
&\quad + \frac{2\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{b,j,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{st} G_{it} G_{bi} \right] + \mathcal{O}(\Psi^5). \tag{C.40}
\end{aligned}$$

Thus, from (C.31), (C.38), (C.39) and (C.40), we conclude that

$$\begin{aligned}
& \mathbb{E} \left[ \frac{1}{N} \sum_{b,j} F''(\mathfrak{X}) \left( \text{Im} \int_{E_1-2}^{E_2-2} G'_{ja} G'_{bj} d\tilde{y} \right) G_{ia} G_{bi} \right] \\
&= \frac{\gamma^2}{(\lambda\gamma v_j - \tau)^2} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^2} \sum_{b,j,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{is} G_{bi} \right] \\
&\quad + \frac{\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{b,j,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{it} G_{bs} G_{ti} \right] \\
&\quad + \frac{\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{b,j,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bt} \tilde{G}_{sj} d\tilde{y} \right) G_{it} G_{bi} \right] \\
&\quad + \frac{2\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F'''(\mathfrak{X})}{N^2} \sum_{b,j,k,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) \left( \text{Im} \int_{E_1-2}^{E_2-2} \hat{G}_{ks} \hat{G}_{tk} d\hat{y} \right) G_{it} G_{bi} \right] \\
&\quad + \frac{\gamma^2}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^2} \sum_{b,j,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} (z + \gamma^2 \tilde{m} - \tau) \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{is} G_{bi} \right] \\
&\quad + \frac{\gamma^2}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^2} \sum_{b,j,s} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) (z + \gamma^2 m - \tau) G_{is} G_{bi} \right] \\
&\quad + \frac{2\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{b,j,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{st} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{it} G_{bi} \right] \\
&\quad + \frac{2\gamma^4}{(\lambda\gamma v_j - \tau)^3} \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{b,j,s,t} \left( \text{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{st} G_{it} G_{bi} \right] + \mathcal{O}(\Psi^5), \tag{C.41}
\end{aligned}$$

which yields, after summing over the index  $j$ , the third “optical theorem” of this appendix,

$$\begin{aligned}
& \gamma^2 \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{b,j,s,t} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{it} G_{bs} G_{ti} \right] \\
& + \gamma^2 \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{b,j,s,t} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bt} \tilde{G}_{sj} d\tilde{y} \right) G_{it} G_{bi} \right] \\
& + 2\gamma^2 \mathbb{E} \left[ \frac{F'''(\mathfrak{X})}{N^2} \sum_{b,j,k,s,t} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \hat{G}_{ks} \hat{G}_{tk} d\hat{y} \right) G_{it} G_{bi} \right] \\
& + \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^2} \sum_{b,j,s} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} (z + \gamma^2 \tilde{m} - \tau) \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{is} G_{bi} \right] \\
& + \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^2} \sum_{b,j,s} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) (z + \gamma^2 m - \tau) G_{is} G_{bi} \right] \\
& + 2\gamma^2 \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{b,j,s,t} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{st} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{it} G_{bi} \right] \\
& + 2\gamma^2 \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{b,j,s,t} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{js} \tilde{G}_{bj} d\tilde{y} \right) G_{st} G_{it} G_{bi} \right] \\
& = \mathcal{O}(\Psi^5). \tag{C.42}
\end{aligned}$$

**C.5. Simplification of (B.18) and Proof of Lemma 8.1.** Recall that  $X_4$  is the sum (over the index  $i$ ) of the terms of order  $\mathcal{O}(\Psi^4)$  given on the right side of (A.85). Subtracting twice (C.22) and  $(2\gamma^2)$ -times (C.30) and (C.42) from (C.5), we obtain

$$\begin{aligned}
& \sum_i (\mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{42}] + \gamma^4 \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}'_{44}] + 4\gamma^4 \mathbb{E}[F'(\mathfrak{X}) \mathcal{X}_{44}]) \\
& + 2\gamma^4 \mathbb{E} \left[ \frac{F''(\mathfrak{X})}{N^3} \sum_{i,j,p,q,s} \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} \tilde{G}_{jp} \tilde{G}_{qj} d\tilde{y} \right) G_{pq} G_{is} G_{si} \right] \\
& = N \mathbb{E} X_4 + \frac{1}{\gamma^2 A_3} \frac{1}{N} \sum_i \frac{1}{(\lambda \gamma v_i - \tau)^2} \mathbb{E}[F'(\mathfrak{X})(z + \gamma^2 m - \tau)] \\
& = -\frac{2A_3}{A_4} N \mathbb{E} X_3 - \frac{1}{\gamma^2 A_4} \frac{1}{N} \sum_i \mathbb{E}[F'(\mathfrak{X}) G_{ii}^2] + \frac{1}{\gamma^4 A_3} \mathbb{E}[F'(\mathfrak{X})(z + \gamma^2 m - \tau)] \\
& = -\frac{2A_3}{A_4} N \mathbb{E} X_3 - \frac{A_2}{\gamma^2 A_4} \mathbb{E}[F'(\mathfrak{X})] + \left( \frac{1}{\gamma^4 A_3} - \frac{2A_3}{\gamma^2 A_4} \right) \mathbb{E}[(z + \gamma^2 m - \tau) F'(\mathfrak{X})], \tag{C.43}
\end{aligned}$$

where we also used (C.7). Plugging (C.43) into (C.10), we conclude that

$$\begin{aligned}
& \frac{1}{N^2} \sum_{i,a,b} (\mathbb{E}[F'(\mathfrak{X}) G_{ia} G_{ab} G_{bi}] + \mathbb{E}[F'(\mathfrak{X}) G_{ia} G_{bb} G_{ai}]) + \frac{1}{N^2} \sum_{i,a,b,j} \mathbb{E} \left[ F''(\mathfrak{X}) \left( \operatorname{Im} \int_{E_1-2}^{E_2-2} G'_{ja} G'_{bj} d\tilde{y} \right) G_{ia} G_{bi} \right] \\
& = N A_1 \mathbb{E}[X_2] + \left( \gamma^{-2} - \frac{2A_3^2}{A_4} \right) N \mathbb{E}[X_3] - \frac{A_2 A_3}{\gamma^2 A_4} \mathbb{E}[F'(\mathfrak{X})] \\
& + \left( \gamma^{-4} - \frac{2A_3^2}{\gamma^2 A_4} \right) \mathbb{E}[(z + \gamma^2 m - \tau) F'(\mathfrak{X})] + \mathcal{O}(\Psi^2). \tag{C.44}
\end{aligned}$$

Finally, combining (C.9) and (C.44), we obtain the estimate (8.2). This completes the proof of Lemma 8.1.

## APPENDIX D

In this last appendix, we prove Lemma 6.6. Recall from (6.25) that we denote  $R(w_{ab}) = F'(\mathfrak{X})G_{ia}G_{bi}$  and that we assumed in Lemma 6.6 that  $i \neq a \neq b \neq j$ .

*Proof of Lemma 6.6.* In (6.6) we defined

$$\mathfrak{X} := N \int_{E_1}^{E_2} \operatorname{Im} m(\widehat{L}_+ + x - 2 + i\eta) dx = N \int_{E_1-2}^{E_2-2} \operatorname{Im} m(\widehat{L}_+ + \widetilde{y} + i\eta) d\widetilde{y}.$$

Also recall that we have abbreviated  $\widetilde{G} \equiv G(\widehat{L}_+ + \widetilde{y} + i\eta)$ . From definition above we see that

$$\frac{\partial \mathfrak{X}}{\partial w_{ab}} = 2 \operatorname{Im} \sum_j \int_{E_1-2}^{E_2-2} \widetilde{G}_{ja} \widetilde{G}_{bj} d\widetilde{y} = \mathcal{O}_\Xi(\Psi).$$

Similarly, we can also show that  $\frac{\partial^2 \mathfrak{X}}{\partial w_{ab}^2} = \mathcal{O}_\Xi(\Psi)$ .

We next consider

$$\begin{aligned} \frac{\partial^2 R(w_{ab})}{\partial w_{ab}^2} &= F'''(\mathfrak{X}) \left( \frac{\partial \mathfrak{X}}{\partial w_{ab}} \right)^2 G_{ia}G_{bi} + F''(\mathfrak{X}) \frac{\partial^2 \mathfrak{X}}{\partial w_{ab}^2} G_{ia}G_{bi} + F''(\mathfrak{X}) \frac{\partial \mathfrak{X}}{\partial w_{ab}} \frac{\partial (G_{ia}G_{bi})}{\partial w_{ab}} \\ &\quad + F'(\mathfrak{X}) \frac{\partial^2 (G_{ia}G_{bi})}{\partial w_{ab}^2}. \end{aligned} \quad (\text{D.1})$$

Clearly the first two terms are of  $\mathcal{O}_\Xi(\Psi^3)$ . Since

$$\frac{\partial (G_{ia}G_{bi})}{\partial w_{ab}} = 2G_{ia}G_{ba}G_{bi} + G_{ib}G_{aa}G_{bi} + G_{ia}G_{bb}G_{ai} = \mathcal{O}_\Xi(\Psi^2),$$

the third term in (D.1) is also  $\mathcal{O}_\Xi(\Psi^2)$ . Finally, in  $(\partial^2/\partial w_{ab}^2)(G_{ia}G_{bi})$  every term contains at least three off-diagonal terms except  $G_{ia}G_{bb}G_{aa}G_{bi}$  and  $G_{ib}G_{aa}G_{bb}G_{ai}$ . We first observe that

$$\mathbb{E}[F'(\mathfrak{X})G_{ia}G_{bb}G_{aa}G_{bi}] = \frac{1}{\lambda\gamma v_a - \tau} \frac{1}{\lambda\gamma v_b - \tau} \mathbb{E}[F'(\mathfrak{X})G_{ia}G_{bi}] + \mathcal{O}_\Xi(\Psi^3).$$

Using the resolvent formula (3.10), we find that

$$G_{ia} = -G_{aa}G_{ii}^{(a)} \left( h_{ia} - \sum_{p,q}^{(a)} h_{ip}G_{pq}^{(a)}h_{qa} \right) = \frac{1}{\lambda\gamma v_a - \tau} \frac{1}{\lambda\gamma v_i - \tau} \left( h_{ia} - \sum_{p,q}^{(a)} h_{ip}G_{pq}^{(a)}h_{qa} \right) + \mathcal{O}_\Xi(\Psi^2).$$

Let

$$\mathfrak{X}^{(a)} := N \int_{E_1}^{E_2} \operatorname{Im} m^{(a)}(\widehat{L}_+ + x - 2 + i\eta) dx = N \int_{E_1-2}^{E_2-2} \operatorname{Im} m^{(a)}(\widehat{L}_+ + y + i\eta) dy,$$

and note that  $\mathfrak{X} - \mathfrak{X}^{(a)} = \mathcal{O}_\Xi(\Psi)$ . Thus, we obtain

$$\begin{aligned} \mathbb{E}_a[F'(\mathfrak{X})G_{ia}G_{bi}] &= \mathbb{E}_a[F'(\mathfrak{X}^{(a)})G_{ia}G_{bi}^{(a)}] + \mathcal{O}_\Xi(\Psi^3) \\ &= \frac{1}{\lambda\gamma v_a - \tau} \frac{1}{\lambda\gamma v_i - \tau} \mathbb{E}_a \left[ F'(\mathfrak{X}^{(a)})G_{bi}^{(a)} \left( h_{ia} - \sum_{p,q}^{(a)} h_{ip}G_{pq}^{(a)}h_{qa} \right) \right] + \mathcal{O}_\Xi(\Psi^3). \end{aligned} \quad (\text{D.2})$$

Since  $i \neq a$ , the first term on the right side of (D.2) vanishes. Therefore we have

$$\mathbb{E}[F'(\mathfrak{X})G_{ia}G_{bi}] = \mathbb{E} \mathbb{E}_a[F'(\mathfrak{X})G_{ia}G_{bi}] = \mathcal{O}_\Xi(\Psi^3).$$

This completes the proof of the Lemma 6.6 and also concludes this last appendix.  $\square$

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